Introduction to Astrometry

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Index

0. Summary

- What is Astrometry?
- General Principles
- Basic Elements of Astrometry
  - References: Time, Space, Units
  - Motion: Linear, Orbital, Rotational
  - Signal Propagation: 1-way, Round-trip
- Mathematical Tools

Astrometry is …

- Quest for Universe through Position/Motion of Celestial Objects
  - Also called: Fundamental Astronomy
  - Astronomy in “Astronomy & Astrophysics”
- Related with
  - Celestial Mechanics
  - Geodesy
  - Special/General Theories of Relativity

General Principles

- 4-dim. Continuous Spacetime
- Law of Causality
- Time Arrow Definiteness
- Deterministic Principle
- Existence of Inertial Frame
- Principles of Relativity

Reference Systems

- RS=Coordinate System + Unit System
- Time Coordinate System
  - Astronomical, Physical, Broadcasting
- Space Coordinate System
  - Horizontal, Equatorial, Ecliptic
  - Solar System Barycentric, Geocentric, Terrestrial(=Earth-Crust-Fixed)
- Unit System: International, Astronomical

Motion

- Cosmic Expansion
- Quasi-Linear Motion: Far Objects
  - Stars, Galaxies, Quasars
- Orbital Motion
  - Quasi-Keplerian: Binary, Comet, Asteroid
  - Complicated: Planet, Satellite, Space Vehicle
- Rotation
  - Earth, Moon, Planet, Satellite, Asteroid, etc.

Signal Propagation

- Electro-Magnetic Wave
  - Visible, IR, Radio, UV, X, Gamma
  - Geometric Optics Approx.: Photon Path
  - Relativistic Treatments
- Cosmic Ray = High Energy Particle
- Gravitational Wave

Mathematical Tools

- Vector Analysis
- Linear Algebra
- Solution of Non-Linear Equation
- Method of Least Squares
- Fourier Analysis
- Numerical Integration of Ordinary Differential Equations

1. Observation

- Global Quantities: Non-Measurable
  - Coordinates, Finite Length
- Local Quantities: Measurable
  - Clock Reading, Angle, Frequency, etc.
- Measuring Methods
  - Passive, Semi-Passive, Active
- New Observing Facilities

Observables

- Clock Reading
  - Epoch: Arrival Time, Emission Time
  - Time Interval = Duration Time
- Angle: Difference in Incoming Vectors
- Others
  - Frequency = Energy
  - Pattern, Code Embedded Artificially

Passive Observation

- Astro-Camera: 2D Angles
  - CCD Array, Video, Photographic Plate
- Theodolite, Meridian Circle: 1D Angle
- Interferometer: Precise 1D Angle
  - VLBI=Very Long Baseline Interferometer
  - Radio, Optical, IR, X-ray, …
- Ground-based VS in-the-Space

Passive Observation (2)

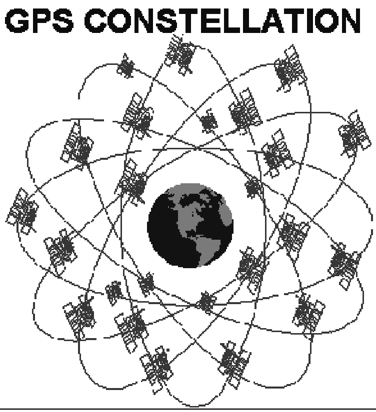
- Detector: Arrival Time, Energy
  - PMT (Photo Multiplier Tube), Photo Diode
  - CCD (Charge Coupled Device), Bubble Chamber
- Clock Reading
  - Event Time: Arrival, Eclipse, Occultation, etc.
- Time Series: Light Curve, Decay Pattern
- Doppler Shift: Radial Velocity
  - Spectrometer, Emission/Absorption Lines

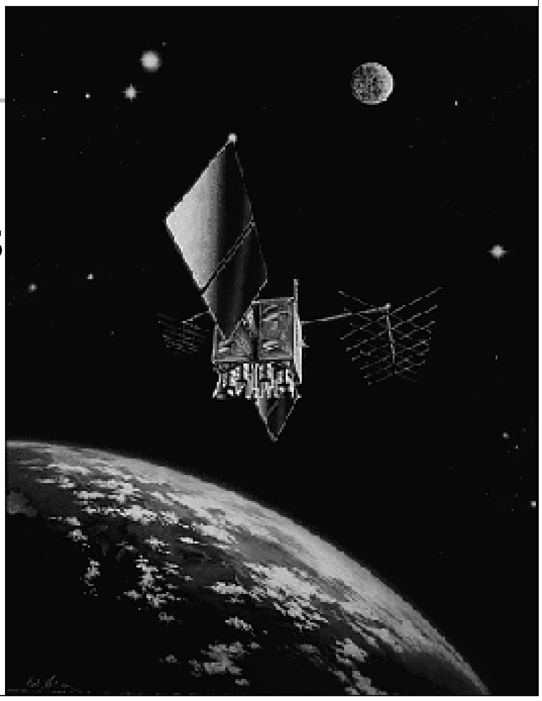
Semi-Passive Obs.

- Doppler Shift
  - Up/Down Link with Artificial Satellite or Space Vehicle
- Integrated Doppler Shits ~ Range Difference
  - NNSS, DORIS/PRARE
- Semi-Passive VLBI: ALSEP, RISE
- Difference Time Obs.: GPS, GLONASS, GALILEO

Global Positioning System

- US DoD
- Flying Atomic Clocks



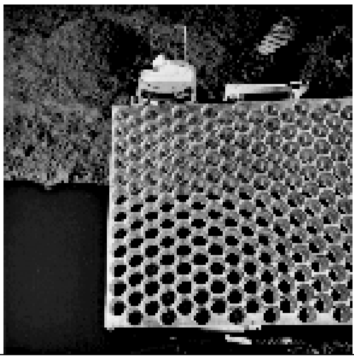


Active Observation

- RADAR Bombing
  - Inner Planets, Near-Earth Asteroids
- Range and Range-Rate (R&RR)
  - Artificial Satellite, Space Vehicle
- Radio Transponding
  - Artificial Satellite, Space Vehicle
- LASER Ranging
  - Artificial Satellite (SLR), Moon (LLR)

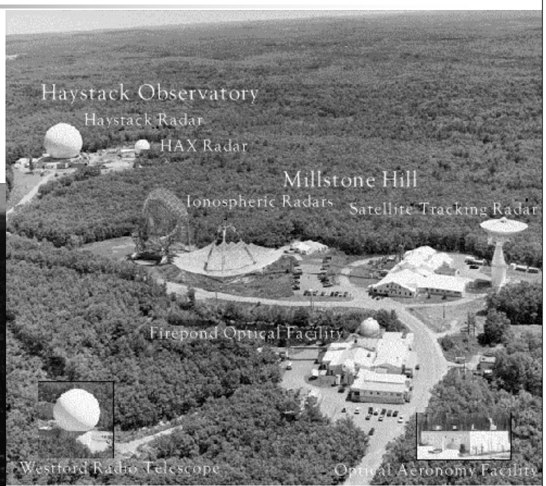
LASER Ranging

- Satellite LR
- Lunar LR
  - 3 Apollo + 2 Luna



RADAR Bombing

- Haystack, MIT
- Arecibo



New Facilities

- Optical/IR Interferometer
  - NPOI, PRIMA/VLTI, SIM, TPF-I
- Orbital Telescope
  - HIPPARCOS, JASMINE, GAIA
- VLBI
  - VLBA, VSOP, VERA, e-VLBI

NPOI

- US Navy Prototype Optical Interferometer
- Flagstaff, Arizona, USA

PRIMA/VLTI

- Phase-Referenced Imaging and Microarcsecond Astrometry
- ESO, Chile
- VLT Outrigger

SIM

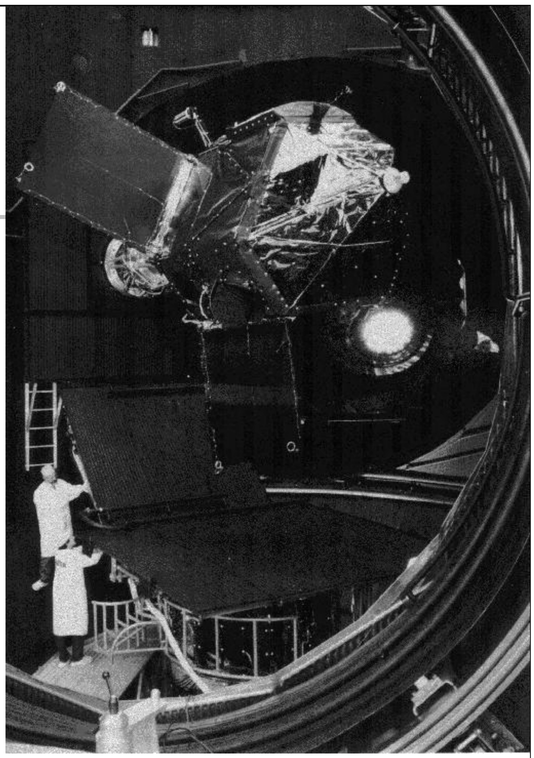
- Space Interferometer Mission, NASA

TPF-I

- Terrestrial Planet Finder-Interferometer

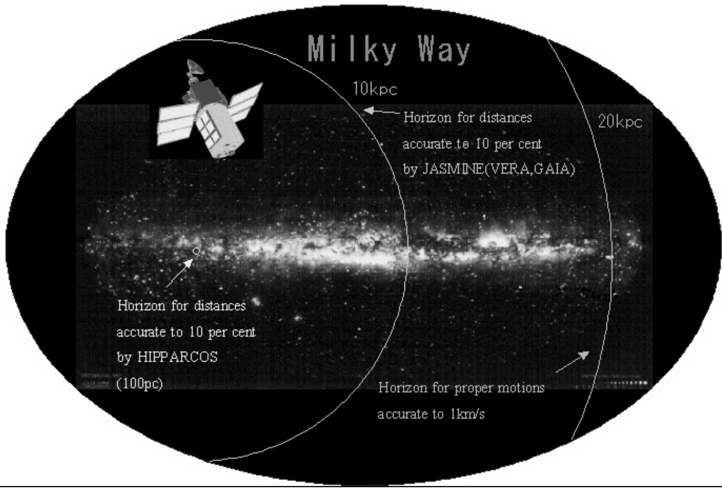
HIPPARCOS

- First Satellite dedicated to Astrometry
- ESA
- Great Achievements



JASMINE

- Japanese Astrometry Satellite Mission

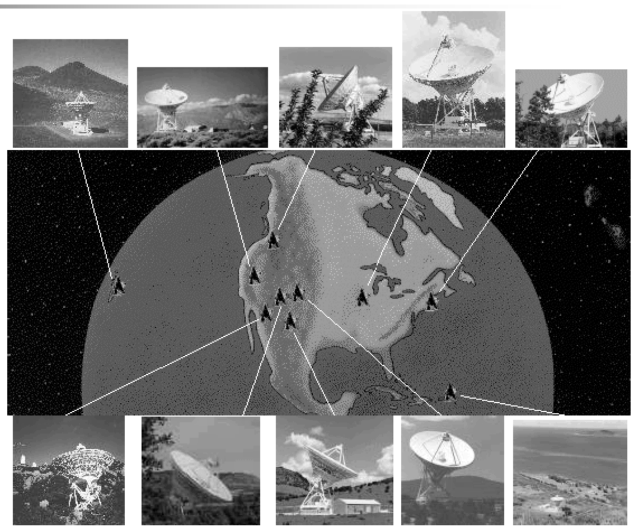


GAIA

- Post-HIPPARCOS
- ESA
- Will be Launched in Summer 2011

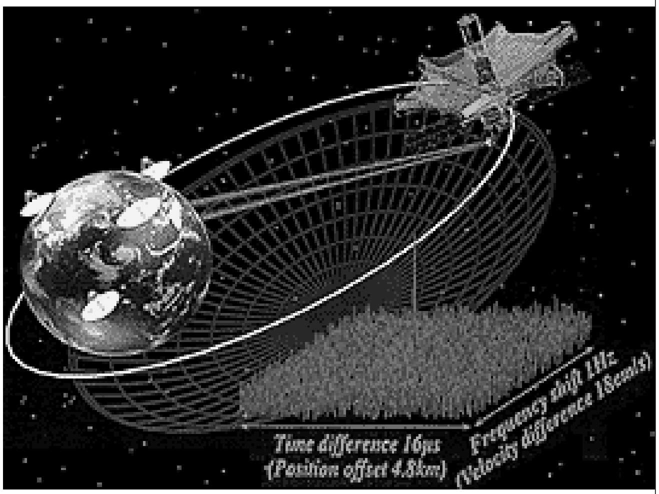
VLBA

- VLBI Array
- 10 Stations in USA
- NRAO, USA



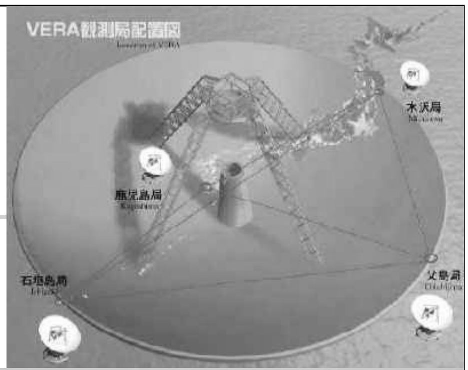
VSOP

- First Space VLBI Mission
- ISAS/NAOJ, Japan



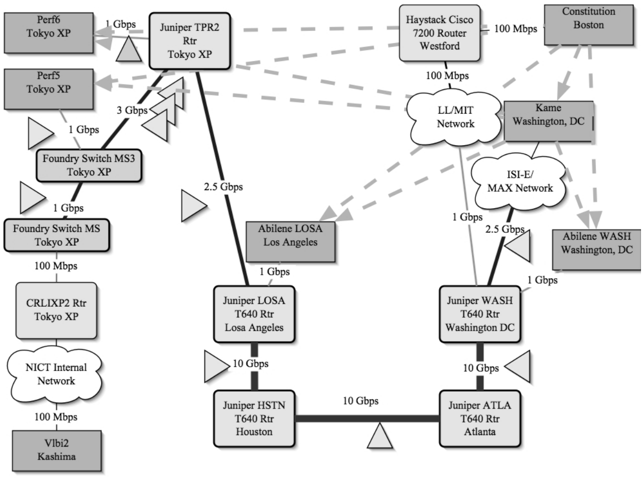
VERA

- Japanese VLBI Array
- 4 Stations
  in Japan
- Two Beam Differenced Observation
- NAOJ



e-VLBI

- Online VLBI via High Speed Internet



2. Time

- Basic Concepts
- Ideal Time Systems
  - Integrated, Dynamical, Broadcasting
- Practical Time Scales
  - Atomic Time, Universal Time
  - Solar System Barycentric/Coordinate Time
- Units and Expression
  - Julian Date

Concepts of Time

- Newtonian Viewpoint
- Absolute Time
- Time Transformation: 1 to 1 $t = f(\tau)$
- Ordering: Chronology
- Precision VS Accuracy
  - Essential Question on Repeatability

Integrated Time System

- Assumption: Constant Duration of A Certain Phenomenon
- Time = Number of Phenomena
- Example
  - Astronomical: Day, Month, Year
  - Mechanical: Pendulum, Spring
  - Physical: Quartz, Molecule, Atom

Dynamical Time System

- Time Argument in Equation of Motion
- Epoch Determined Inversely from Observation
- Example
  - Mean Longitude of the Sun
  - L(T)= $279^\circ 41' 48''.04 + 129602769''.13T + 1''.089T^2$
  - Epehemeris Time: ET=T(L)

Broadcasting Time System

- Time Signals in the Air: JJY, TV, NTT
  - NTP: TS on Computer Network
  - GPS Time: TS from GPS Satellites
- Standard Time
  - Time Zone: 15 degree = 1 Hour
- Japan Standard Time: JST
  - JST = UTC + 9 h

Atomic Time

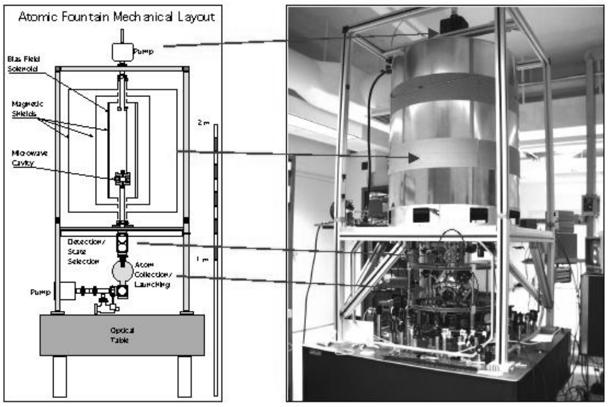
- Definition of SI Second: CGPM (1967)
  - 9192631770 Periods
  - Specific Radiation from Cesium 133
- International Atomic Time: TAI
  - Steered by BIPM (Paris)
  - Hundreds of Cesium Atomic Clocks
    + Several Hydrogen Maser Clocks
  - Relative Precision: 15-16 Digits

Cesium Atomic Clock

- HP/Agilent 5071A

Atomic Fountain Clock

Fountain Clock Hardware



Hydrogen Maser Clock

Universal Time

- Dynamical TS based on Earth Rotation
  - UT = GMT (Greenwich Mean Solar Time)
  - 3 Variations: UT0, UT1, UT2
  - Monitored by IERS
- UTC (Coordinated Universal Time)
- Leap Second
  - Secular Deceleration of Earth Rotation

Solar System Dynamical Time

- Official TS of IAU (1984-1991)
  - General Relativistic Effects Considered
  - TDB: SS Barycentric Dynamical Time
  - TDT: Terrestrial Dynamical Time
  - Unit Adjustment: <TDB> = <TDT>
- TDT = TAI+32.184s

Solar System Coordinate Time

- Official TS of IAU (1991-)
  - No Unit Adjustment
  - TCB: SS Barycentric Coordinate Time
  - TCG: Geocentric Coordinate Time
  - TT: Terrestrial Time
  - TT = TDT = TAI+32.184s
- TCB-TCG: Time Ephemeris
  - Harada and Fukushima (2003)

Time Units

- 1 day=24 hours=1440 min.=86400 s
- Julian Century: jc, Julian Year: jy
  - 1 jc = 100 jy = 36525 days
- Besselian Year = Mean Solar Year = 365.2421897… days
- ms, μs, ns, ps, fs, …
- Speed of Light: c = 299792458 m/s

Time Expression

- Year, Month, Day, Hour, Minute, Second
  - Day of Week, Day of Year
- Julian Date: JD
  - J2000.0 = 12 O'clock, Jan. 1st, 2000 = JD2451545.0
- Modified Julian Date: MJD
  - MJD = JD – 2400000.5

Julian Date

- From (Y, M, D, h, m, s) to JD
  - L=int((M-14)/12);
  - I=1461*(Y+4800+L);
  - J=367*(M-2-12*L);
  - K=int((Y+4900+L)/100);
  - N=int(I/4)+int(J/12)-int((3*K)/4)

Julian Date (2)

- JD0=N+D-32075;
- JD1=JD0-0.5;
- JD2=h/24.0+m/1440.0+s/86400.0;
- JD=JD1+JD2 or JD = (JD1,JD2)

Julian Date (3)

- From JD to (Y, M, D, h, m, s)
  - JD0=int(JD-0.5); JD1=JD0-0.5;
  - L=JD0+68569;
  - N=int((4*L)/146097);
  - K=L-int((146097*N+3)/4);
  - I=int(4000*(K+1))/1461001);
  - P=K-int((1461*I)/4)+31;

Julian Date (4)

- J=int((80*P)/2447);
- D=P-int((2447*J)/80); Q=int(J/11);
- M=J+2-12*Q; Y=100*(N-49)+I+Q;
- JD2=JD-JD1;
- h=int(JD2*24)
- m=int(JD2*1440-h*60);
- s=JD2*86400-h*3600-m*60;

Day of Week

- I=JD0-7*int((JD0+1)/7)+2;
- I: 1,2,3,4,5,6,7
- I=1: Sunday

3. Space

- Space Coordinate and Unit
- Spacial Coordinate Transformation
  - Rectangular, Spherical, Spheroidal
- Inertial Coordinate System
  - Parallel Transport of Coordinate Origin, Rotation around Origin
- Velocity and Acceleration

Spatial Coordinates

- Rectangular (x, y, z)
- Spherical (r, θ, λ) (r, ϕ, λ)
- Spheroidal (φ, λ, h)

Spherical Coordinate

- Horizontal
  $r,\ (z, A);(a, A);(\mathrm{Alt}, \mathrm{Az});(\mathrm{El}, \mathrm{Az})$
- Ecliptic $r\,,\beta\,,\lambda$
- Equatorial $\pi\,,\alpha\,,\delta$
- Galactic $\pi\,,b\,,\ell$

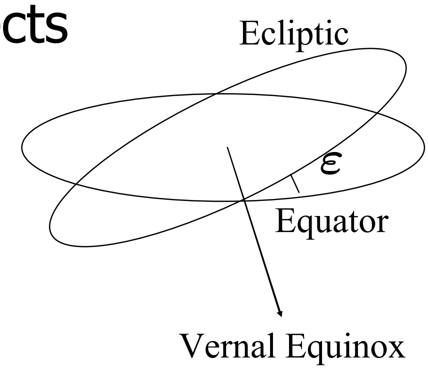
Horizontal Coordinate

- Radius: r, Zenith Distance: z
- Altitude (Angle)
  - a = Alt = El = 90 deg – z
- Azimuth(al Angle) : A = Az, Left-Handed

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \sin z \cos A \\ -\sin z \sin A \\ \cos z \end{pmatrix} = r \begin{pmatrix} \cos a \cos A \\ -\cos a \sin A \\ \sin a \end{pmatrix}$$

Ecliptic Coordinate

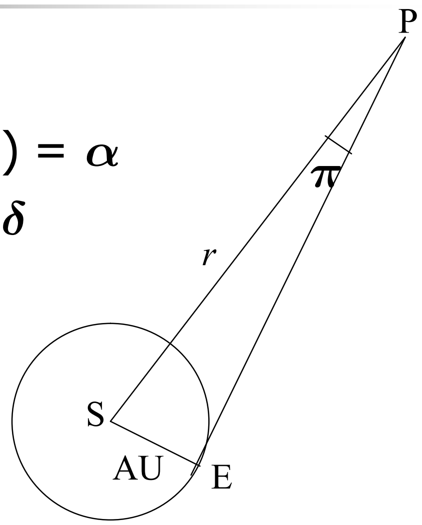
- Ecliptic ~ Mean Earth Orbit
- For Solar System Objects
- Obliquity of Ecliptic: ε
  - Radius: r
  - Longitude: λ
  - Latitude: β



Equatorial Coordinate

- Basic Representation
  - Right Ascension (R.A.) = α
  - Declination (Decl.) = δ
  - (Annual) Parallax: π

$$\pi = \sin^{-1}\left(\frac{\text{AU}}{r}\right)$$



Angle Units

- Radian: rad
  - 180 deg = π rad
- Degree: deg = °
- Minute of Arc: min = arc minute = '
- Second of Arc: second = arc second = " = arcsec = as

Angle Units (2)

- 1 deg = 60 arcmin = 3600 arcsec
- 180 deg = π rad
- 1 arcsec ~ 4.848 μrad
- 20 arcsec ~ 0.1 mrad: Aberration
- 0.001 arcsec = milli-arcsec: mas
- 0.000001 arcsec = micro-arcsec: μas

Length Units

- SI meter: Defined via SI Second
  - Speed of Light: c = 299792458 m/s
- Astronomical Unit (of Length): AU
  - Rough: Mean Radius of Earth Orbit
  - Rigorous: AU = $c\tau$, τ = 499.00478353… s
- Parsec (pc), Light Year (ly)
  - 1 pc = AU/sin 1" ~ 30.9 Pm ~ 3.26 ly
  - 1 ly = c x 1 jy ~ 9.5 Pm

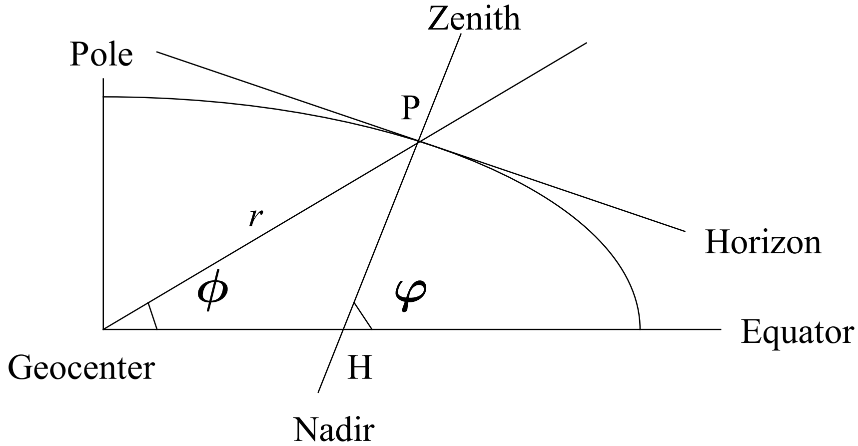
Spheroidal Coordinate

- Geographic Latitude: φ
- Longitude: λ
- Height from Reference Ellipsoid: h

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho_N \cos\varphi \cos\lambda \\ \rho_N \cos\varphi \sin\lambda \\ \rho_Z \sin\varphi \end{pmatrix}$$

Geographic Latitude

- Geocentric Latitude: ϕ
- Geographic/Geodetic Latitude: φ



Spheriodal Coord. (2)

- Ellipsoid Normal: N
  - =Radius of Curvature ACROSS Meridian

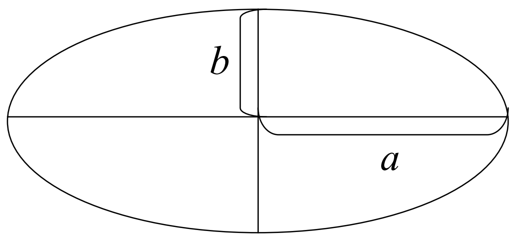
$$\rho_N = N + h,\ \rho_Z = \left(1 - e^2\right)N + h$$

$$N = \frac{a}{d},\ d = \sqrt{1 - e^2 \sin^2 \varphi}$$

Ellipse

- Semi-Major Axis: a
- Semi-Minor Axis: b

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1$$



Flattening Factor

- Flattening Factor: f
- Eccentricity: e, Complimentary Ecc.: e_c

$$f \equiv \frac{a-b}{a}, \; e_c \equiv \frac{b}{a} = \sqrt{1-e^2} = 1-f$$

$$e^2 \equiv \frac{a^2-b^2}{a^2} = 2f - f^2$$

Spherical to Rectangular

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \sin\theta\cos\lambda \\ \sin\theta\sin\lambda \\ \cos\theta \end{pmatrix} = r \begin{pmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{pmatrix}$$

Rectangular to Spherical

$$r = \sqrt{x^2 + y^2 + z^2},\ p = \sqrt{x^2 + y^2}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \text{atan2}\,(p, z),$$

$$\phi = \sin^{-1}\left(\frac{z}{r}\right) = \text{atan2}\,(z, p),$$

$$\lambda = \text{atan2}\,(y, x)$$

Spheroidal to Rectangular

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho_N \cos\varphi \cos\lambda \\ \rho_N \cos\varphi \sin\lambda \\ \rho_Z \sin\varphi \end{pmatrix}$$

$$\rho_N = N + h, \rho_Z = \left(1 - e^2\right)N + h$$

$$N = \frac{a}{d}, d = \sqrt{1 - e^2 \sin^2 \varphi}$$

Rectangular to Spheroidal

- Difficult Inverse Problem
- Easy: Longitude $\lambda = \mathrm{atan2}(y, x)$
- Eliminating Longitude

$$\begin{cases} (N+h)\cos\varphi = p \equiv \sqrt{x^2 + y^2} \\ \left(N\left(1-e^2\right)+h\right)\sin\varphi = z \end{cases}$$

- -> Latitude Equation

Latitude Equation

- After Elimination of h

$$p \sin\varphi - z\cos\varphi = \frac{C\sin\varphi\cos\varphi}{\sqrt{1 - e^2\sin^2\varphi}}$$

where $C = ae^2$

Latitude Equation (2)

- Variable Transformation $t = \cot\varphi$
- Transformed Equation
- Derivation and Solution

$$f(t) \equiv zt + \frac{Ct}{\sqrt{g+t^2}} - p = 0$$

$$\text{where}\ \ g \equiv 1 - e^2$$

Derivation of Lat. Eq.

$$\sin\varphi = \frac{1}{\sqrt{1+t^2}}, \cos\varphi = \frac{t}{\sqrt{1+t^2}}$$

$$\therefore \frac{p}{\sqrt{1+t^2}} - \frac{zt}{\sqrt{1+t^2}} = \frac{C}{\sqrt{1-e^2\frac{1}{1+t^2}}}\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}}$$

$$p - zt = \frac{Ct}{\sqrt{\left(1-e^2\right)+t^2}}$$

Solution of Lat. Eq.

- Localization (Northern Hemisphere) $0 \le z$
- Variable Domain after Localization

$$f(0) = -p \le 0,\ \ f(+\infty) \approx zt + C \ge 0$$
$$\therefore 0 \le t \le +\infty$$

- Newton Method
- Initial Guess

$$t_0 = \frac{p}{z + C/\sqrt{g}}$$

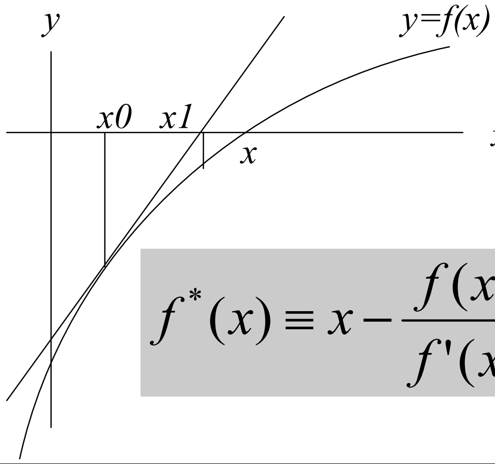
Newton Method

- Effective to Solve Nonlinear Eq.

$$f(x) = 0$$

- Essence
  - Linearization
- Newton Iteration

$$x \to f^*(x)$$

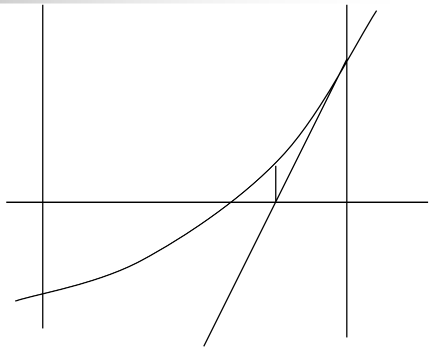


$$f^*(x) \equiv x - \frac{f(x)}{f'(x)}$$

Newton Method (2)

- Quadratic Convergence
  - Doubling Effective Digits
- Fast but Unstable
- Slow when Multiple Roots
- Key Points
  - Bracketing to Assure Uniqueness
  - Selecting Stable Starters

Stable Starter



- Bracketing

  $x_L < x < x_R$

- Assumption 1

  $f(x_L) < 0 < f(x_R)$

- Assumption 2

  $x_L < x < x_R \rightarrow f'(x) > 0, f''(x) > 0$

- Stable Starter: Upper Bound of Solution

Application to Lat. Eq.

- Preparation

$$f(t) \equiv zt + \frac{Ct}{\sqrt{g+t^2}} - p$$

$$f(0) = -p \leq 0 \leq f(+\infty)$$

$$f'(t) = z + \frac{Cg}{\left(\sqrt{g+t^2}\right)^3} > 0$$

$$f''(t) = \frac{-3Cgt}{\left(\sqrt{g+t^2}\right)^5} < 0$$

Application (2)

- Newton Iteration

$$f^*(t) \equiv t - \frac{f(t)}{f'(t)} = \frac{p\left(\sqrt{g+t^2}\right)^3 - Ct^3}{z\left(\sqrt{g+t^2}\right)^3 + Cg}$$

- Stable Starter: Lower Bound of Solution

$$t_0 = 0 \rightarrow f^*(0) = \frac{p}{z + C/\sqrt{g}}$$

Velocity & Acceleration

- Velocity = Variation of Position
- Acceleration = Variation of Velocity
- Jerk = Variation of Acceleration

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}, \quad \mathbf{a} = \frac{d^2\mathbf{x}}{dt^2}, \quad \mathbf{j} = \frac{d^3\mathbf{x}}{dt^3}$$

Velocity in Spherical CS

- Vector Representation

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}} = \left(\frac{\partial \mathbf{x}}{\partial r}\right)\frac{\mathrm{d}r}{\mathrm{dt}} + \left(\frac{\partial \mathbf{x}}{\partial \phi}\right)\frac{\mathrm{d}\phi}{\mathrm{dt}} + \left(\frac{\partial \mathbf{x}}{\partial \lambda}\right)\frac{\mathrm{d}\lambda}{\mathrm{dt}}$$

$$= v_r \mathbf{e}_r + v_\phi \mathbf{e}_\phi + v_\lambda \mathbf{e}_\lambda$$

- Component Representation

$$v_r = \frac{\mathrm{d}r}{\mathrm{dt}},\ v_\phi = r\frac{\mathrm{d}\phi}{\mathrm{dt}},\ v_\lambda = r\cos\phi\frac{\mathrm{d}\lambda}{\mathrm{dt}}$$

Coordinate Triad in Spherical CS

$$\mathbf{e}_r \equiv \left(\frac{\partial \mathbf{x}}{\partial r}\right) = \begin{pmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{pmatrix}$$

$$\mathbf{e}_\phi \equiv \frac{1}{r}\left(\frac{\partial \mathbf{x}}{\partial \phi}\right) = \begin{pmatrix} -\sin\phi\cos\lambda \\ -\sin\phi\sin\lambda \\ \cos\phi \end{pmatrix}$$

$$\mathbf{e}_\lambda \equiv \frac{1}{r\cos\phi}\left(\frac{\partial \mathbf{x}}{\partial \lambda}\right) = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$

Velocity in Spheroidal CS

- Vector Representation

$$\mathbf{v} = \left(\frac{\partial \mathbf{x}}{\partial h}\right)\frac{\mathrm{d}h}{\mathrm{dt}} + \left(\frac{\partial \mathbf{x}}{\partial \varphi}\right)\frac{\mathrm{d}\varphi}{\mathrm{dt}} + \left(\frac{\partial \mathbf{x}}{\partial \lambda}\right)\frac{\mathrm{d}\lambda}{\mathrm{dt}} = v_h \mathbf{e}_h + v_\varphi \mathbf{e}_\varphi + v_\lambda \mathbf{e}_\lambda$$

- Component Representation

$$v_h = \frac{\mathrm{d}h}{\mathrm{dt}},\ v_\varphi = \rho_M \frac{\mathrm{d}\varphi}{\mathrm{dt}},\ v_\lambda = \rho_N \cos\varphi \frac{\mathrm{d}\lambda}{\mathrm{dt}}$$

$$\rho_M = M + h, \quad M = \frac{a\left(1-e^2\right)}{\left(\sqrt{1-e^2\sin^2\varphi}\right)^3}$$

Coordinate Triad in Spheroidal CS

$$\mathbf{e}_h \equiv \left(\frac{\partial \mathbf{x}}{\partial h}\right) = \begin{pmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{pmatrix}$$

$$\mathbf{e}_\varphi \equiv \frac{1}{\rho_M}\left(\frac{\partial \mathbf{x}}{\partial \varphi}\right) = \begin{pmatrix} -\sin\varphi\cos\lambda \\ -\sin\varphi\sin\lambda \\ \cos\varphi \end{pmatrix}$$

$$\mathbf{e}_\lambda \equiv \frac{1}{\rho_N \cos\varphi}\left(\frac{\partial \mathbf{x}}{\partial \lambda}\right) = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$

Radius of Curvature in Spheroidal CS

- RC Across Meridian: East-West Direction

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}} \qquad \rho_N = N + h, \rho_Z = \left(1 - e^2\right)N + h$$

- RC In Meridian: North-South Direction

$$M = \frac{a\left(1 - e^2\right)}{\left(\sqrt{1 - e^2 \sin^2 \varphi}\right)^3} \qquad \rho_M = M + h$$

$$\frac{\mathrm{d}\left(\rho_N \cos\varphi\right)}{\mathrm{d}\varphi} = \rho_M \sin\varphi, \quad \frac{\mathrm{d}\left(\rho_Z \sin\varphi\right)}{\mathrm{d}\varphi} = \rho_M \cos\varphi$$

5. Coordinate System

- 4-Dim. Coordinate System (CS)
  = Time System + Spatial CS
- Inertial CS
- Accelerated CS and Inertial Force
  - Rotating CS: Coriolis F, Centrifugal F
- Coordinate Transformation
  - Galilean CS, Rigid-Body Rotation

Inertial Coordinate System

- CS where Law of Inertia Holds
- Newton's Law of Inertia
  - No Force -> Linear Motion
- Galilei's Principle of Relativity
  - Law of Physics is Invariant at Any ICS
- Parallel Transport of Coordinate Origin
  - ICS to ICS

Parallel Transport of Coordinate Origin

- Galactic Center in Quasar-Rest Frame
  - Cosmic Expansion
- Local Standard of Rest in Galactic CS
  - Local Standard of Rest (LSR) = Solar System Barycenter
  - Feature of Local Motion: Oort's Constant

Parallel Transport of Coordinate Origin (2)

- Geocenter in Solar System Barycentric CS
  - Planetary Ephemeris
- Averaged Crust in Geocentric CS
  - Earth Rotation
- Observer in Terrestrial CS
  - Fixed to Earth Surface (= Averaged Crust)
  - Surface Motion (Aircraft, Ship, Car, etc)

Ephemeris and Almanac

- Numerical Table on Complicated Motion
  - Orbit: Planets, Satellites, Asteroids
  - Rotation: Planets, Satellites
- Astronomical Almanac (US+UK)
- Japanese Ephemeris
- NASA/JPL DE series, DE413/408
  - Most Precise, Machine Callable

Spatial Coordinate Transformation

- General Transf. $x_k \leftarrow X_j = X_j\left(x_k, t\right)$

- Taylor Expansion w.r.t. New Coordinates

$$
\begin{aligned}
X_j &= X_j\left(0,t\right) + \sum_{k=1}^{3} \frac{\partial X_j}{\partial x_k}\left(0,t\right) x_k + \cdots \\
&= A_j\left(t\right) + \sum_{k=1}^{3} B_{jk}\left(t\right) x_k + \cdots
\end{aligned}
$$

Linear Transformation

- General Affine Transformation

$$\mathbf{X} = \mathbf{A}(t) + \mathrm{B}(t)\mathbf{x}$$

- Static: 12-Parameter Transformation

$$\mathbf{X} = \mathbf{A} + \mathrm{B}\mathbf{x}$$

Coefficient Matrix

$$\mathrm{B} = \mathrm{D} + \mathrm{S} + \Theta$$

- Scaling: Diagonal Component

$$\mathrm{D}_{jk} = 0 \quad \text{if} \quad j \neq k$$

- Shear: Non-Diagonal, Symmetric

$$\mathrm{S}_{jk} = \mathrm{S}_{kj} \qquad \left(\mathrm{S}_{jk} = 0 \;\; \text{if} \;\; j = k\right)$$

- Infinitesimal Rotation: Asymmetric

$$\Theta_{jk} = -\Theta_{kj}$$

7-Parameter Transformation

- CT between Similar Two CS
  - Isotropic Scaling
  - Origin Shift
  - Rotation

$$\mathbf{X} = \mathbf{X}_0 + \left(s\mathrm{I} + \Theta\right)\mathbf{x}$$

- Ex.: Transf. among Geocentric CSs
  - World Geod. System (ITRFnn, GRS80)
  - Tokyo Datum and JGD 2000

6. Motion of Celestial Bodies

- Rest: Quasar
- Linear: Most of Stars
- Rotation: Earth, the Moon, Satellite
- Kepler: Binaries
- Quasi-Kepler: Asteroid, Satellite
- Complicated: Planet, Space Vehicle

Resting Body

- Quasar: Practically Being Rest
- Position Expression
  - Epoch t_0
  - Mean Place at Epoch (α_0, δ_0)
  - Parallax at Epoch π_0
- Quasar Catalogs: IAU, ICRFnn

Linear Motion

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0 (t - t_0)$$

- Different Treatment for Radial Comp.
- Proper Motion = Linear Motion on Celestial Sphere

$$\mathbf{x} = r \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix} \qquad \begin{pmatrix} r \\ \alpha \\ \delta \end{pmatrix} \cong \begin{pmatrix} r_0 \\ \alpha_0 \\ \delta_0 \end{pmatrix} + \begin{pmatrix} V_R \\ \mu_\alpha \\ \mu_\delta \end{pmatrix} (t - t_0)$$

Star Catalog

- Epoch, Mean Place, and Parallax
- Propor Motion (μ_α, μ_δ)
- Radial Velocity V_R
- Astrophysical Information
  - Luminosity, Color, Variability, etc.
- Astrometric Star Catalogs
  - HIPPARCOS, FKn, PPM, AGKn

7. Rotation

- Rotation = Orthogonal Transformation
  - Infinitesimal Rotation: Vector Product
  - Finite Rotation: Orthogonal Matrix
- Euler’s Theorem
- Fundamental Rotation
- Angular Velocity

Orthogonal Transformation

- Distance Invariant in Euclidean Space

$$(\Delta \mathbf{X})^2 = (\Delta \mathbf{x})^2$$

- Rotation: A Linear Transformation

$$\Delta \mathbf{X} = \mathrm{R} \Delta \mathbf{x}$$

- Orthogonality

$$(\mathrm{R}\Delta \mathbf{x})^2 = \Delta \mathbf{x}^{\mathrm{T}} \mathrm{R}^{\mathrm{T}} \mathrm{R} \Delta \mathbf{x} = (\Delta \mathbf{x})^2$$

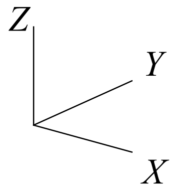
$$\therefore \mathrm{R}^{\mathrm{T}}\mathrm{R} = \mathrm{I} \ \text{ or } \ \mathrm{R}^{-1} = \mathrm{R}^{\mathrm{T}}$$

Finite Rotation

- Expression: Matrix, Spinor, Quarternion
- Rotational Operation: Matrix Product
- Rotation Matrix = Coordinate Triad

= Trio of Orthonormal Basis

$$\mathrm{R} = \begin{pmatrix} \mathbf{e}_X & \mathbf{e}_Y & \mathbf{e}_Z \end{pmatrix}^{\mathrm{T}}$$



Euler's Theorem

- Any Finite Rotation = Triple Product of Fundamental Rotation Matrices

$$\mathrm{R} = \mathrm{R}_{ijk}(\alpha, \beta, \gamma) \equiv \mathrm{R}_k(\gamma)\mathrm{R}_j(\beta)\mathrm{R}_i(\alpha)$$

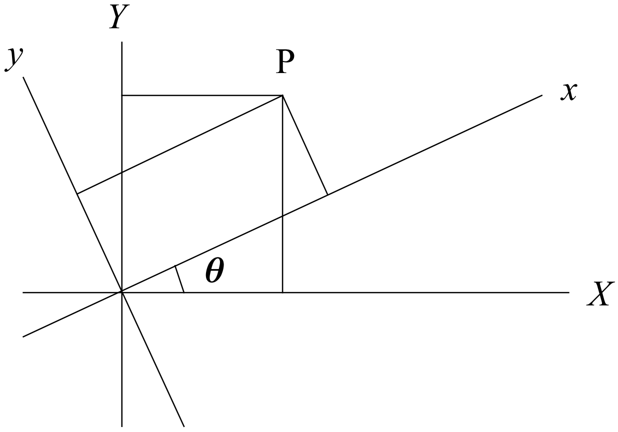
$$\left(\mathrm{R}_{ijk}(\alpha, \beta, \gamma)\right)^{-1} = \mathrm{R}_{kji}(-\gamma, -\beta, -\alpha)$$

- Euler Angles = 3 Fundamental Rotation Angles

Fundamental Rotation Operation

- Rotate around z-axis by the angle θ

$$R_3(\theta) = R_z(\theta)$$



Fundamental Rotation Operation (2)

- Rotation Around j-th Axis by θ

$$\mathrm{R}_j(\theta)$$

- Reverse Rotation

$$\left(\mathrm{R}_j(\theta)\right)^{-1} = \mathrm{R}_j(-\theta)$$

Fundamental Rotation Matrix

$$\mathrm{R}_3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Ex.: Ecliptic-Equatorial Transf. $\mathrm{R}_1(\varepsilon)$
- Obliquity of Ecliptic ε

Fundamental Rotation Matrix (2)

- Small Angle Approximation

$$\mathrm{R}_3(\theta) \cong \mathrm{I} + \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{I} - \theta\, \mathbf{e}_3 \times$$

$$\therefore \prod_j \mathrm{R}_j(\theta_j) \cong \mathrm{I} - \left(\sum_j \theta_j \mathbf{e}_j \right) \times$$

Euler Rotation

- Combinations of Euler Angles: 3x2x2=12
- 3-1-3 (=X) Convention
  - Most Popular, the Euler Angles
  - Used in Classic Rotational Dynamics

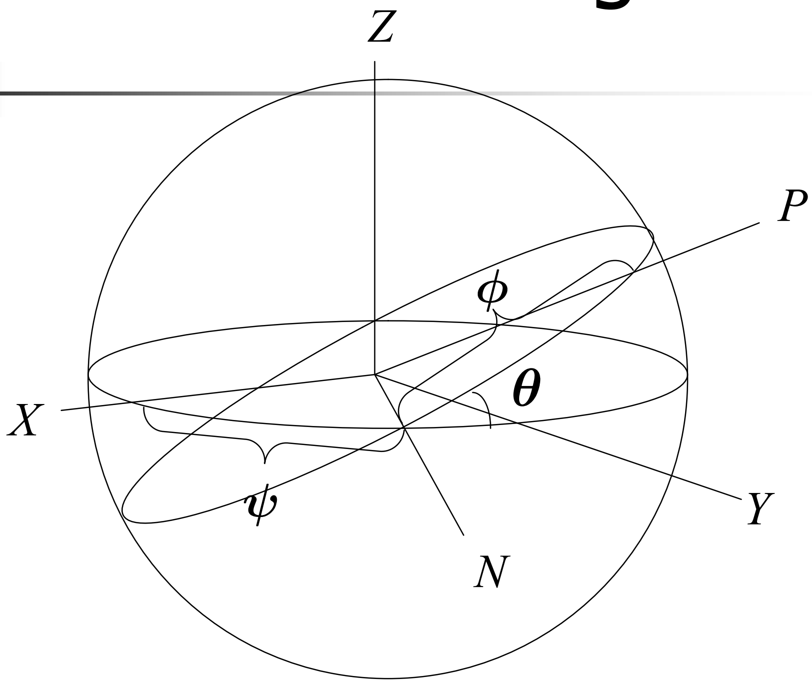
$$\mathrm{R}_{313}(\psi,\theta,\phi)=\mathrm{R}_3(\phi)\mathrm{R}_1(\theta)\mathrm{R}_3(\psi)$$

3-1-3 Rotation Matrix

$$\mathrm{R}_{313}(\psi,\theta,\phi)=\begin{pmatrix} C_\psi C_\phi - S_\psi C_\theta S_\phi & C_\psi S_\phi + S_\psi C_\theta C_\phi & S_\psi S_\theta \\ -S_\psi C_\phi - C_\psi C_\theta S_\phi & -S_\psi S_\phi + C_\psi C_\theta C_\phi & C_\psi S_\theta \\ S_\theta S_\phi & -S_\theta C_\phi & C_\theta \end{pmatrix}$$

$$=\begin{pmatrix} \cos\psi\cos\phi-\sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi+\sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi-\cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi+\cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

3-1-3 Euler Angles



Weak Point of 3-1-3 Convention

- Degeneracy in Small Angles

$$\mathrm{R}_{313}(\psi,\theta,\phi) \cong \mathrm{I} - \begin{pmatrix} \theta \\ 0 \\ \phi+\psi \end{pmatrix} \times$$

- Recipe
  - Use 3-2-1 and Other Convention with All Different Indices

3-2-3 Convention

- Alias: Y-Convention, Ex.: Precession

$$\mathrm{P} = \mathrm{R}_{323}\left(-\zeta_A, \theta_A, -z_A\right)$$

- Screw: Rotation Around Fixed Direction

$$\mathrm{R}_{323}\left(\lambda, \varphi, \chi\right) = \mathrm{I} + \left(\sin\chi\right)\mathbf{n}\times + \left(1 - \cos\chi\right)\mathbf{n}\times\mathbf{n}\times$$

$$\mathbf{n} = \begin{pmatrix} \sin\varphi\cos\lambda \\ \sin\varphi\sin\lambda \\ \cos\varphi \end{pmatrix}$$

Other Conventions

- 1-3-1: Nutation

  $$\mathrm{N} = \mathrm{R}_{131}\left(\varepsilon_A, -\Delta\psi, -\left(\varepsilon_A + \Delta\varepsilon\right)\right)$$

- 2-1-3: Polar Motion + Sidereal Rotation

  $$\mathrm{WS} = \mathrm{R}_{312}\left(\Theta, -y_p, -x_p\right)$$

- 1-2-3: Aerodynamics, Attitude Control
  - One of Most Desirable Conventions

Rotation and Velocity Transformation

$$\mathbf{X} = \mathrm{R}\mathbf{x} \quad \Rightarrow$$

$$\begin{aligned}\mathbf{V} &= \mathrm{R}\mathbf{v} + \frac{\mathrm{dR}}{\mathrm{dt}}\mathbf{x} \\ &= \mathrm{R}\mathbf{v} - \Omega\mathbf{x} \\ &= \mathrm{R}\mathbf{v} - \boldsymbol{\omega} \times \mathbf{x}\end{aligned}$$

Angular Velocity

$$\boldsymbol{\omega} = \sum_j \frac{\mathrm{d}\theta_j}{\mathrm{dt}} \mathbf{e}_j$$

$$\because \mathrm{R} = \prod_j \mathrm{R}_j\left(\theta_j\right) \cong \mathrm{I} - \left(\sum_j \theta_j \mathbf{e}_j\right)\times$$

$$\frac{\mathrm{dR}}{\mathrm{dt}} = -\left[\sum_j \left(\frac{\mathrm{d}\theta_j}{\mathrm{dt}}\right)\mathbf{e}_j\right]\times = -\boldsymbol{\omega}\times = -\Omega$$

Infinitesimal Rotation

- 3D Anti-Symmetric Matrix ~ Axial Vector

$$\Theta = \begin{pmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{pmatrix} \qquad \boldsymbol{\theta} = \begin{pmatrix} \theta_x \\ \theta_y \\ \theta_z \end{pmatrix}$$

- True Meaning of Vector Product

$$\Theta \Delta \mathbf{x} = \boldsymbol{\theta} \times \Delta \mathbf{x}$$

Small Angle Rotation

$$\mathrm{R}_{123}(\alpha,\beta,\gamma)=\mathrm{R}_3(\gamma)\mathrm{R}_2(\beta)\mathrm{R}_1(\alpha)$$

$$=\begin{pmatrix} C_\gamma C_\beta & C_\gamma S_\beta S_\alpha + S_\gamma C_\alpha & -C_\gamma S_\beta C_\alpha + S_\gamma S_\alpha \\ -S_\gamma C_\beta & -S_\gamma S_\beta S_\alpha + C_\gamma C_\alpha & S_\gamma S_\beta C_\alpha + C_\gamma S_\alpha \\ S_\beta & -C_\beta S_\alpha & C_\beta C_\alpha \end{pmatrix}$$

$$\cong \mathrm{I} - \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}\times$$

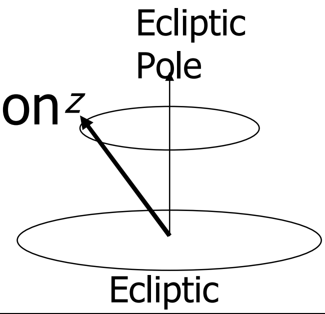
8. Earth Rotation

- Base of Coordinate Transformation between Geocentric and Terrestrial Coordinate System
- Sidereal Rotation (S) … Rotation Angle UT1
- Motion of Figure Axis
  - Quasi-Diurnal: Polar Motion = Wobble (W)
  - Others: Precession (P) + Nutation (N)
- Matrix Representation

$$R = WSNP$$

Precession + Nutation

- Figure Axis Motion (Other than Wobble)
- 2 Components in Ecliptic CS
  - Longitude, Obliquity
- Precession=Very Long Periodic Motion
  - 50 arcsec/y, ~26000y Period
- Nutation=Other Periodic Motion
  - 18.6y, 0.5y, 9.3y, etc
- New Model Soon Appears



Precession

- Discovery: Hipparchus (~150BC)
- Old Model: IAU1976
  - Lieske et al. (1976, A&A)
  - Dynamics: Newcomb's Theory
  - Correction of Planetary Masses
  - Adding Geodesic Precession
- Theory: in Ecliptic CS
- Formula: in Equatorial CS

Precession (2)

$$P = R_{323}\left(-\zeta_A, \theta_A, -z_A\right)$$

- 3 Precession Angles in Equatorial CS

$$\begin{pmatrix} \zeta_A \\ \theta_A \\ z_A \end{pmatrix} = \begin{pmatrix} 2306.2181 \\ 2004.3109 \\ 2306.2181 \end{pmatrix} T + \begin{pmatrix} 0.30188 \\ -0.42665 \\ 1.09468 \end{pmatrix} T^2 + \begin{pmatrix} 0.017998 \\ -0.041833 \\ 0.018203 \end{pmatrix} T^3$$

  - Unit: 1 arcsec
  - T =(JD-2451545.0)/36525

Precession (3)

- Approximation of Precession Matrix

$$\mathrm{P} \cong \begin{pmatrix} 1 & -\phi_A & -\theta_A \\ \phi_A & 1 & 0 \\ \theta_A & 0 & 1 \end{pmatrix}$$

$$\phi_A \equiv \zeta_A + z_A$$

- Correction in R.A. and Decl.

$$\Delta_P \alpha = \phi_A + \theta_A \tan\delta \sin\alpha, \quad \Delta_P \delta = \theta_A \cos\alpha$$

Precession (4)

- Approximation of Precession Angle

$$\phi_A \cong m_P T, \theta_A \cong n_P T$$

- Precession (Speed) in R.A. and Decl.

$$m_P = 4612.4362''/\text{jy}, \quad n_P = 2004''.3109''/\text{jy}$$

- Approximate Correction Formula

$$\Delta_P \alpha \cong \left(m_P + n_P \tan\delta \sin\alpha\right)T,$$
$$\Delta_P \delta \cong \left(n_P \cos\alpha\right)T$$

Nutation

- Discovery: Bradley (1747)
- Old Model: IAU1980
  - Seidelmann et al. (1981, CM)
  - Rigid Earth: Kinoshita (1977, CM)
  - Non-Rigidity: Wahr (1981, GJRAS)
- Mean Obliquity (Lieske et al., 1976)

$$\varepsilon_A = 23°26'21''.448 - 46''.8150T$$
$$- 0''.00059T^2 + 0''.001813T^3$$

Nutation (2)

- Matrix Representation

$$\mathrm{N} = \mathrm{R}_{131}\left(\varepsilon_A, -\Delta\psi, -\left(\varepsilon_A + \Delta\varepsilon\right)\right)$$

- Nutation in Longitude $\boldsymbol{\Delta\psi}$
- Nutation in Obliquity $\boldsymbol{\Delta\varepsilon}$
- Analytic Expression

$$\begin{pmatrix} \Delta\psi \\ \Delta\varepsilon \end{pmatrix} = \sum_k \begin{pmatrix} \psi_k \sin A_k \\ \varepsilon_k \cos A_k \end{pmatrix}, \quad A_k = \sum_{j=1}^{5} n_j \Omega_j$$

Delauney Angles

- Main 5 Angles in Nutation Theory
  - Mean Anomaly of Moon ℓ
  - Mean Anomaly of Sun ℓ'
  - Mean Argument of Latitude of Moon F
  - Mean Elongation $D \equiv L - L'$
  - Mean Longitude of Ascending Node of Moon
- Details: Seidelmann et al. (1981) Ω

Rough Approx. of Nutation

- Precision: 0.1 arcsec
- Unit: 1 arcsec

$$\begin{pmatrix}\Delta\psi \\ \Delta\varepsilon\end{pmatrix} = \begin{pmatrix}-17.2\sin\Omega \\ 9.2\cos\Omega\end{pmatrix} + \begin{pmatrix}-1.3\sin 2L' \\ 0.6\cos 2L'\end{pmatrix} + \begin{pmatrix}0.2\sin 2\Omega \\ -0.1\cos 2\Omega\end{pmatrix}$$

$$+\begin{pmatrix}-0.2\sin 2L \\ 0.1\cos 2L\end{pmatrix} + \begin{pmatrix}0.1\sin \ell' \\ 0\end{pmatrix} + \begin{pmatrix}0.1\sin \ell \\ 0\end{pmatrix} + \cdots$$

Approx. Nutation

- Approximation of Nutation Matrix

$$\mathrm{N} \cong \begin{pmatrix} 1 & -\Delta\mu & -\Delta\nu \\ \Delta\mu & 1 & -\Delta\varepsilon \\ \Delta\nu & \Delta\varepsilon & 1 \end{pmatrix}$$

- Nutation in R.A. and Decl.

$$\Delta\mu = \Delta\psi \cos\varepsilon_A, \Delta\nu = \Delta\psi \sin\varepsilon_A$$

Approx. Nutation (2)

- Correction in R.A. and Decl.

$$\Delta_N \alpha = \Delta\mu + \tan\delta\left(\Delta\nu \sin\alpha - \Delta\varepsilon\cos\alpha\right),$$
$$\Delta_N \delta = \Delta\nu\cos\alpha + \Delta\varepsilon\sin\alpha$$

Sidereal Rotation

- Almost Uniform Quasi-Diurnal Rotation
  - Ω_0 = 7.2921150(1) x 10^{-5} radian/s
- Angular Rotation = 360 degree/Sidereal Day ~ 365.2422.../366.2422... Rot./Day
- Greenwich Apparent Sidereal Time (GAST) Θ

$$\mathrm{S} = \mathrm{R}_3\left(\Theta\right)$$

Deviation from Uniform Rotation

- UTC → UT1 → GMST → GAST
  - DUT1 = UT1-UTC: Unpredictable
  - GMST = $GMST_0$ + r UT1 + ...
  - Ratio of Sidereal/Universal Time: r
  - r ~ 1.0027379...
  - GAST = GMST + $\Delta\psi$ cos ε + ...
- Length of Day (LOD) = $2\pi/\Omega$

Polar Motion = Wobble

- Slow Motion of Pole Viewed on Earth
  - Symbol: (x_p, y_p), Size: 0.1 arcsec ~ 30m
  - Periods: Annual, Chandler (~14 month)
- Unpredictable = To be Monitored

$$W = R_2\left(-x_p\right)R_1\left(-y_p\right)$$

EOP

- Earth Orientation Parameters
  - DUT1, LOD, x_p, $y_{p,}$, Pole Offsets
  - Old Terms: Earth Rotation Parameters (ERP)
- Pole Offset = Error in Prec./Nut. Theory
- International Earth Rotation Service (IERS)
  - Since 1984, Joint Service of IAU and IUGG
  - Homepage: http://www.iers.org/

9. Keplerian Motion

- Solution of Two-Body Problem
- Gravitational Constant

  $\mu = G(M + m)$

  $$\frac{d^2\mathbf{x}}{dt^2} = \frac{-\mu}{r^3}\mathbf{x}$$

- Orbital Element = 6 Constants
  - Shape of Orbit a, e
  - Orientation of Orbital Plane Ω, I, ω
  - Location in Orbit T

Unit of Mass

- SI Unit of Mass: kg
- Astronomical Unit: Solar Mass M_S
- Universal Constant of Gravitation: G
- Observable = GM = Gravitational Constant of Central Body
  - Heliocentric GC = Sun's GM GM_S
  - Geocentric GC = Earth's GM GM_E

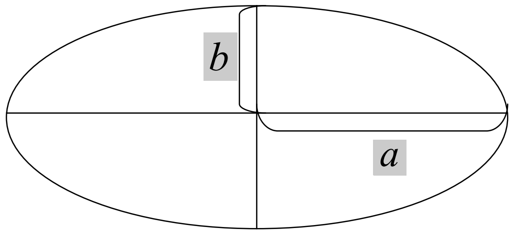
Orbital Elements

- Semi-Major Axis: a
- Orbital Eccentricity: e
- Longitude of Ascending Node: Ω
- Orbital Inclination: I
- Argument of Pericenter: ω
- Time of Pericenter Passage: T

Ellipse

- Semi-Major Axis: a
- Semi-Minor Axis: b

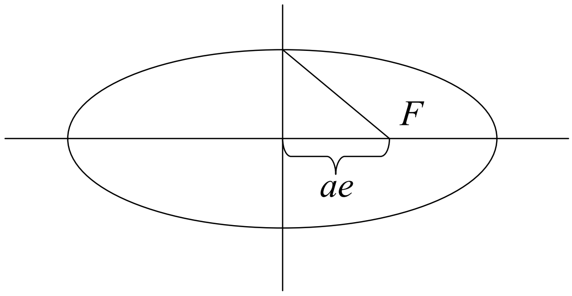
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Orbital Eccentricity

- Eccentricity: e, Complimentary Ecc.: e'

$$e \equiv \sqrt{\frac{a^2 - b^2}{a^2}}, \;\; e' \equiv \frac{b}{a} = \sqrt{1 - e^2}$$



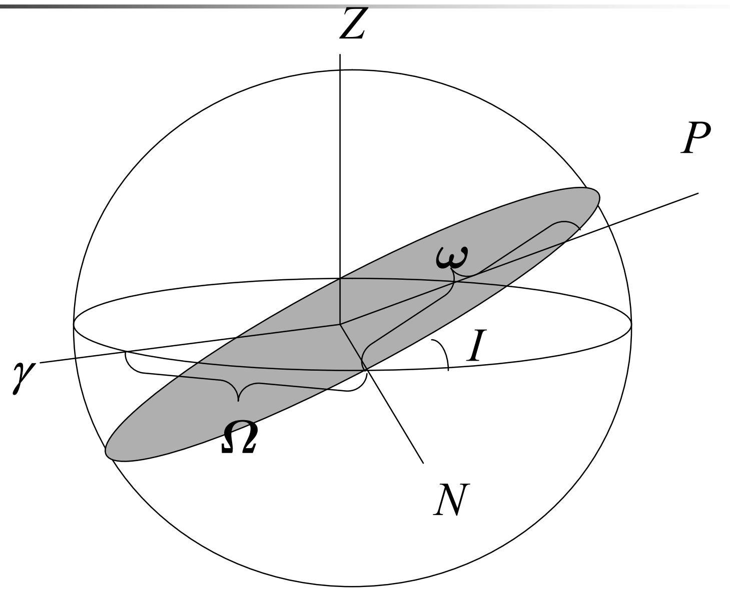
Orbital Plane

- 3-1-3 Euler Angles of Orbital Plane

$$\mathrm{R}_{313}(\Omega, I, \omega) = \mathrm{R}_3(\omega)\mathrm{R}_1(I)\mathrm{R}_3(\Omega)$$

- 3 Important Direction Vectors
  - Origin of Longitude: X-axis
  - Ascending Node: N
  - Pericenter: P

Orbital Plane (2)



Keplerian Orbit

- Elliptical: $e < 1$
  - Planet, Satellite, Binary
- Parabola: $e = 1$
  - Good Approximation of Comet Orbit
- Quasi-Parabola: $e \sim 1$
  - Comet, Peculiar Asteroids
- Hyperbolic: $e > 1$
  - Space Vehicle, Close Encounter

Element to Position and Velocity (Elliptic)

- Solve (Elliptic) Kepler's Equation

$$E - e\sin E = n(t - T)$$

- Speed of Ecc. Anomaly E $\dot{E} = \dfrac{n}{1 - e\cos E}$
- PV in Orbital Plane

$$\begin{cases} \xi = a(\cos E - e) \\ \eta = b\sin E \end{cases} \qquad \begin{cases} \dot{\xi} = -a\dot{E}\sin E \\ \dot{\eta} = b\dot{E}\cos E \end{cases}$$

Element to Position and Velocity (Parabolic)

- Solve Barker's Eq.

= Parabolic Kepler's Eq.

$$\tau + \frac{\tau^3}{3} = \sqrt{\frac{\mu}{2q^3}}\left(t - T\right)$$

- Speed of τ

$$\dot{\tau} = \frac{1}{1+\tau^2}\sqrt{\frac{\mu}{2q^3}}$$

- PV in Orbital Plane

$$\begin{cases} \xi = q\left(1-\tau^2\right) \\ \eta = 2q\tau \end{cases} \qquad \begin{cases} \dot{\xi} = -2q\tau\dot{\tau} \\ \dot{\eta} = 2q\dot{\tau} \end{cases}$$

Element to Position and Velocity (Hyperbolic)

- Solve (Hyperbolic) Kepler's Equation

$$e \sinh F - F = n(t - T)$$

- Speed of F

$$\dot{F} = \frac{n}{e \cosh F - 1}$$

- PV in Orbital Plane

$$\begin{cases} \xi = a(e - \cosh F) \\ \eta = b \sinh F \end{cases}$$

$$\begin{cases} \dot{\xi} = -a\dot{F} \sinh F \\ \dot{\eta} = b\dot{F} \cosh F \end{cases}$$

Element to PV (2)

- Reverse Euler Rotation

$$\left(\mathbf{x},\mathbf{v}\right)=\mathrm{R}_{313}\left(-\omega,-I,-\Omega\right)\begin{pmatrix}\xi & \dot{\xi}\\ \eta & \dot{\eta}\\ 0 & 0\end{pmatrix}$$

Kepler's Equation

- First Transcendenal Equation in History
- Elliptic $$E - e\sin E = M$$
- Parabolic $$\tau + \frac{\tau^3}{3} = M_P$$
- Hyperbolic $$e\sinh F - F = M_H$$

Elliptic Kepler's Eq.

- Eccentric Anomaly: E
- Mean Anomaly: M
- Kepler's 3rd Law
- True Anomaly: f

$$E - e\sin E = M$$

$$M = n\left(t - T\right)$$

$$\mu = n^2 a^3$$

$$\begin{cases} \xi = a\left(\cos E - e\right) = r\cos f \\ \eta = b\sin E = r\sin f \end{cases}$$

Solution of Kepler's Eq.

$$f(E) \equiv E - e\sin E - M = 0$$

- Domain Reduction

$$-\infty < M < \infty \Rightarrow 0 \le M < \pi \quad \Rightarrow 0 \le E < \pi$$

- Newton Method

$$E \rightarrow f^*(E)$$

$$f^*(E) \equiv E - \frac{f(E)}{f'(E)} = \frac{M - e(E\cos E - \sin E)}{1 - e\cos E}$$

Stable Starter of Newton Merthod

- Stability Theory of Newton Method

$$f(0) \le 0 < f(\pi),$$
$$f'(E) > 0, f''(E) > 0$$

- Upper Bound as Stable Starter $f^*(E_0)$
- Examples

$$E_0 = \min\left(f^*(0), f^*\left(\frac{\pi}{2}\right), f^*(\pi) \right)$$
$$= \min\left(\frac{M}{1-e}, M+e, \frac{M+\pi e}{1+e} \right)$$

Perturbed Keplerian Orbit

- Element = Slow Function of Time

$$\Lambda \equiv (a, e, I, \Omega, \omega, T) = \Lambda(t)$$

- Perturbation Theory
- Polynomials + Fourier Series

$$\Lambda = \Lambda_0 + \Lambda_1 t + \Lambda_2 t^2 + \cdots$$
$$+ \sum_k \left(C_k \cos \nu_k t + S_k \sin \nu_k t \right)$$

Complicated Orbit

- Eq. of Motion $\frac{d^2\mathbf{x}}{dt^2} = \frac{-\mu}{r^3}\mathbf{x} + \cdots$
- Solution
  - Numerical: Numerical Integration
  - Analytical: Perturbation Theory
- Parameters Estimation by Fitting Solution to Obs. Data
- Result: Astronomical Ephemeris

Astronomical Ephemeris

- Numerical: DE (NASA/JPL, USA)
- Analytical: VSOP/ELP (BdL, France)
- DE: Available through NAOJ/ADAC
  - Software (Fortran/C) + Binary Files
  - DE408: BC10000-AD10000, UNIX/Win/Mac
  - PV of Sun, Moon, and 9 Major Planets
- Whole Solar System Bodies: HORIZONS
  - http://ssd.jpl.nasa.gov/

10. Signal Propagation

- Geometric Optics Approximation
- Basic: One-way Propagation
- Application: Multi-way Prop.
- Light Direction: Aberration & Parallax
- Doppler Shift
- Propagation Delay

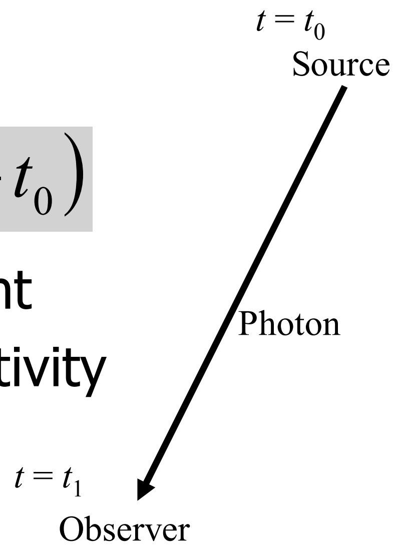
One-Way Propagation

- Photon: Linear Motion

$$\mathbf{X}(t) = \mathbf{X}_0 + \mathbf{V}_0(t - t_0)$$

- Constant Speed of Light
- Special Theory of Relativity

$$|\mathbf{V_0}| = c$$

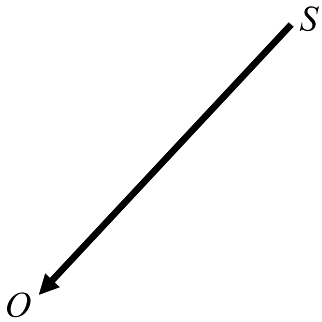


Passive Observables

- Arrival Epoch t_1
- Incoming Direction $\mathbf{d}_1$
- Observed Wavelength λ_1

Eq. of Light Time

- Within Solar System
- Departure Epoch t_0
- Arrival Epoch t_1
- Light Time = Duration



$$\tau \equiv t_1 - t_0$$

- Equation to Solve LT $c\tau = R_{10}\left(\tau\right)$

Eq. of Light Time (2)

- Diff. in Departure/Arrival Position

$$\mathbf{x}_1 - \mathbf{x}_0 = \mathbf{V}\left(t_1 - t_0\right)$$

- Evaluate Magnitude of Diff. Vector

$$R_{10} = V\tau$$

- Assume that Source/Observer Motions are Known $\mathbf{x}_S(t), \mathbf{x}_O(t)$

Eq. of Light Time (3)

$$\mathbf{x}_0 = \mathbf{x}_S\left(t_0\right), \mathbf{x}_1 = \mathbf{x}_O\left(t_1\right), R_{10}\left(\tau\right) \equiv \left|\mathbf{R}_{10}\left(\tau\right)\right|,$$
$$\mathbf{R}_{10}\left(\tau\right) = \mathbf{x}_O\left(t_1\right) - \mathbf{x}_S\left(t_0\right) = \mathbf{x}_1 - \mathbf{x}_S\left(t_1 - \tau\right)$$

- Use Constant Speed of Light

$$V = c$$

- Final: Equation of Light Time

$$c\tau = R_{10}\left(\tau\right)$$

Eq. of Light Time (4)

$$f(\tau) \equiv c\tau - R(\tau) = 0$$

- Newton Method

$$\tau \to f^*(\tau)$$

- Correction Formula

$$f^*(\tau)' \equiv \tau - \frac{f(\tau)}{f'(\tau)} = \frac{R(\tau) - \tau R'(\tau)}{c - R'(\tau)}$$

Eq. of Light Time (5)

- Initial Guess: Infinite Speed of Light
- One Newton Correction

$$\tau^{(1)} \equiv f^{*}(0) = \frac{R_{SO}}{c - V_{SO}}$$

$$R_{SO} = \left|\mathbf{x}_S(t_1) - \mathbf{x}_1\right|, \ V_{SO} = \frac{\left(\mathbf{v}_S(t_1) - \mathbf{v}_1\right)\cdot\left(\mathbf{x}_S(t_1) - \mathbf{x}_1\right)}{R_{SO}}$$

- Next Stage: General Relativity Needed

Light Direction

$$\mathbf{d} = \frac{-\mathbf{V}_1}{\left|\mathbf{V}_1\right|} = \frac{-\mathbf{R}_{10}}{R_{10}}$$

- Aberration: Effect of Observer's Velocity
- Parallax: Effect of Observer's Position
- Periods: Annual, Diurnal, Monthly, etc.
- Correction for Light Time: MUST within Solar System

Aberration

- Bradley (1727)
- Finiteness of Speed of Light
  - Ex.: Raindrops Trails on Side Window
- Vector Expression of Aberration

$$\mathbf{d}' = \frac{-\left(\mathbf{V}_1 - \mathbf{v}_1\right)}{\left|\mathbf{V}_1 - \mathbf{v}_1\right|} = \frac{\mathbf{d} + \mathbf{v}_1 / c}{\left|\mathbf{d} + \mathbf{v}_1 / c\right|} \approx \mathbf{d} + \frac{\mathbf{v}_1 - \left(\mathbf{d} \cdot \mathbf{v}_1\right)\mathbf{d}}{c}$$

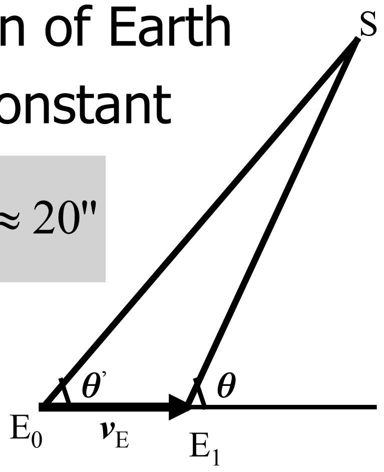
Annual Aberration

- Effect of Orbital Motion of Earth
- (Annual) Aberration Constant

$$\kappa \equiv \frac{v_E}{c} \approx \frac{30\,\text{km/s}}{3\times 10^5\ \text{km/s}} = 10^{-4} \approx 20''$$

- Angle Expression

$$\theta' \cong \theta - \kappa \sin\theta$$



Annual Aberration (2)

- Ecliptic Coordinate System is Useful
- Approximation Formula

$$\begin{cases} \Delta_A \beta \approx \kappa \sin\beta \sin(L-\lambda) \\ (\cos\beta)\Delta_A \lambda \approx -\kappa \cos(L-\lambda) \end{cases}$$

- Mean Longitude of Sun: L
- Aberration Ellipse $\left(\frac{(\cos\beta)\Delta_A\lambda}{\kappa}\right)^2 + \left(\frac{\Delta_A\beta}{\kappa\sin\beta}\right)^2 = 1$

Diurnal Aberration

- Effect of Earth Rotation
- Equatorial CS is Useful
- Diurnal Aberration Constant

$$\kappa' \equiv \frac{R_E \omega_E}{c} \approx \frac{480\mathrm{m/s}}{3\times 10^8\,\mathrm{m/s}} = 1.6\times 10^{-6} \approx 0.3''$$

- Approx. Formula
$$\begin{cases} \Delta'_A \delta \approx \kappa' \cos\phi \sin\delta \sin(\Theta - \alpha) \\ (\cos\delta)\Delta'_A \alpha \approx -\kappa' \cos\phi \cos(\Theta - \alpha) \end{cases}$$
  - Sidereal Rotation Angle: Θ, Geoc. Lat.: ϕ

Parallax

- Bessel (1838): 81 Cygni
- Deviation of Observer's Position from its Mean Value
  - Ex.: Direction Difference between Right/Left Eye's View
- Vector Expression of Parallax

$$\mathbf{d} = \frac{\mathbf{R}}{R} = \frac{\mathbf{x}_0 - \mathbf{x}_1}{\left|\mathbf{x}_0 - \mathbf{x}_1\right|} = \frac{\mathbf{d}_0 - \mathbf{x}_1 / r_0}{\left|\mathbf{d}_0 - \mathbf{x}_1 / r_0\right|} \approx \mathbf{d}_0 - \frac{\mathbf{x}_1 - \left(\mathbf{d}_0 \cdot \mathbf{x}_1\right)\mathbf{d}_0}{r_0}$$

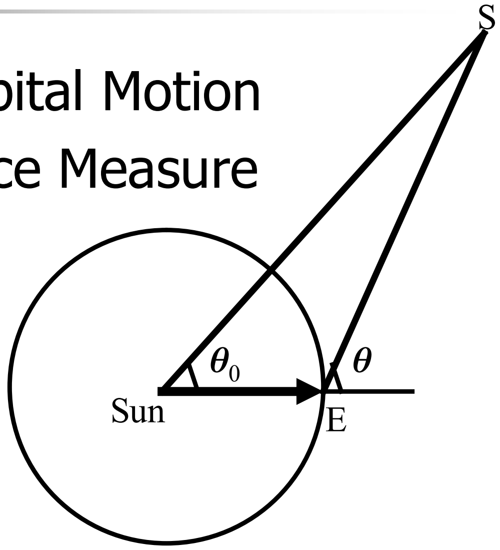
Annual Parallax

- Effect of Earth Orbital Motion
- Alternative Distance Measure

$$\pi \approx \frac{1\,\mathrm{AU}}{r_0}$$

- Angle Expression

$$\theta \approx \theta_0 + \pi \sin \theta_0$$



Annual Parallax (2)

- Approximation Formula in Ecliptic CS

$$\begin{cases} \Delta_\pi \beta \approx \pi \sin \beta_0 \cos(L - \lambda_0) \\ (\cos \beta_0) \Delta_\pi \lambda \approx \pi \sin(L - \lambda_0) \end{cases}$$

- Note: 90 deg Phase Diff. from Aberration
- Parallactic Ellipse

$$\left(\frac{(\cos \beta_0) \Delta_\pi \lambda}{\pi} \right)^2 + \left(\frac{\Delta_\pi \beta}{\pi \sin \beta_0} \right)^2 = 1$$

Diurnal Parallax

- Effect of Earth Radius: Moon, Artificial Sat.
- Approximation Formula in Equatorial CS

$$\begin{cases} \Delta'_{\pi}\,\delta \approx \pi' \cos\phi \sin\delta \cos(\Theta - \alpha) \\ (\cos\delta)\Delta'_{\pi}\,\alpha \approx \pi' \cos\phi \sin(\Theta - \alpha) \end{cases}$$

- Yet Another Distance Measure: Horizontal Parallax

$$\pi' \equiv \sin^{-1}\left(\frac{R_E}{r}\right) \approx \left(\frac{R_E}{1\mathrm{AU}}\right)\pi \approx 4\times 10^{-5}\,\pi$$

Doppler Shift

- Classic (= Non-Relativistic) Approximation

$$z \equiv \frac{\lambda_1 - \lambda_0}{\lambda_0} = \frac{\left(\mathbf{v}_0 - \mathbf{v}_1\right)\cdot \mathbf{d}}{c}$$

- Outgoing Object = Red Shift
- Incoming Object = Blue Shift

Doppler Shift (2)

- Similar to Aberration
  - Again Aberration Constant
- Annual Doppler Shift

$$\Delta z \approx \kappa \cos \beta \sin\left(L - \lambda\right)$$

- Diurnal Doppler Shift

$$\Delta z' \approx \kappa' \cos \phi \cos \delta \sin\left(\Theta - \alpha\right)$$

Propagation Delay

- Vacuum Delay: General Relativity
  - Color Independent
- Medium Delay
  - Eminent in Longer Wavelength (Radio, etc.)
  - Inter-Galactic/Stellar Matter, Solar Corona
  - Ionosphere, Troposphere
  - Atomosphere

Wavelength-Dependent Delay

$$\Delta\tau(f) = A + \frac{B}{f} + \frac{C}{f^2} + \cdots$$

- Elimination by Multiple Wave Observation
  - Geodetic VLBI: S-band + X-band
  - GPS: L1-band + L2 band
  - Space Vehicle: Up-Link + Down-Link
- Use of Empirical Model: Not-Good
  - Solar Corona, Ionosphere, Troposphere

Delay Model

- Solar Corona
  - Muhleman and Anderson (1981)

$$\Delta\tau_{\text{CORONA}} = \frac{40.3}{cf^2}\int N_e ds \qquad N_e = \frac{A}{r^6} + \cdots$$

- Troposphere (Chao 1970): Zenith Angle, z

$$\Delta\tau_{\text{TROP}} = \frac{7\text{ns}}{\cos z + \dfrac{0.0014}{\cot z + 0.045}}$$

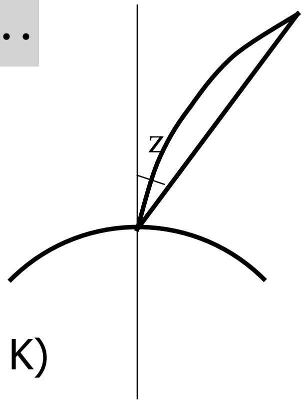
Refraction

- Variation in Incidental Zenith Angle
  - Saastamoinen (1972)

$$\Delta z = a \tan z + b \tan^3 z + \cdots$$

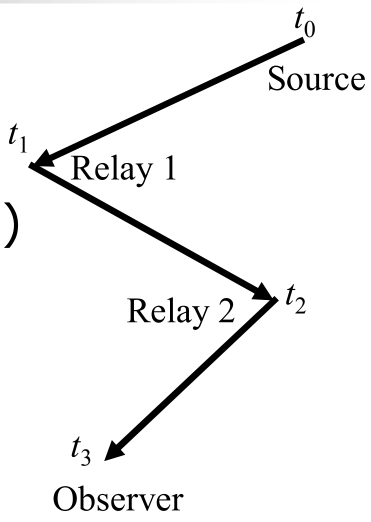
$$a = 16''.271 \left(\frac{P - 0.156 P_W}{T} \right)$$

P: Atmosph. Pressure (Unit: hP)
P_W: Water Vapor Pr. (Unit: hP)
T: Absolute Temperature (Unit: K)



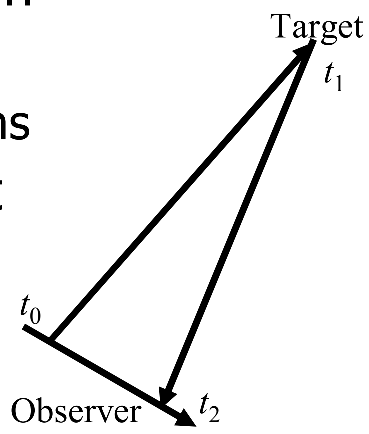
Multi-Way Propagation

- Appl. of One-Way Prop.
- Series of Eq. of Light Time
  - Ex.: Three-Way (t_3, t_2, t_1, t_0)
- Delay in Relay
  - Optical: 0
  - Radio: Constant
    - Specific to Transponder



Round-Trip Propagation

- Typical Active Observation
- Observable
  - Emission/Reception Epochs
- Useful even when Target Motion is Unknown
- Sum of One-way Prop.
- Cancellation at 1st Order



Round-Trip Light Time

- Approximation of Reflection Epoch

$$t_1 = \frac{t_0 + t_2}{2} + O\left(\left(\frac{v}{c}\right)^2 \left(\frac{t_2 - t_0}{2}\right)\right)$$

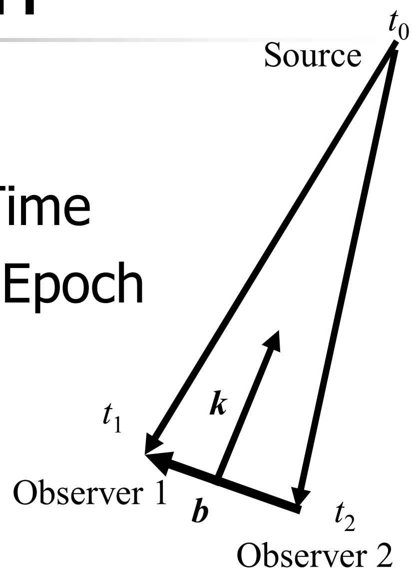
- Approximation of Distance at Reflection

$$R_{SO} = \frac{c\left(t_2 - t_0\right)}{2}\left[1 + O\left(\frac{v}{c}\right)^2\right], R_{SO} = \left|\mathbf{x}_S\left(t_1\right) - \mathbf{x}_O\left(t_1\right)\right|$$

Quasi-Simultaneous Propagation

- Almost Same Arrival
- Pair of Eq. of Light Time
- Difference in Arrival Epoch

$$\tau = t_2 - t_1$$



Interference Observation Equation

- Difference in Eq. of Light Time
- Alias: VLBI Observation Eq.

$$c\tau = -\mathbf{b} \cdot \mathbf{k}$$

- Baseline Vector ***b***
- Midpoint Direction ***k***

$$\mathbf{b} = \mathbf{x}_2 - \mathbf{x}_1$$

$$\mathbf{k} = \frac{\mathbf{x}_0 - \left(\mathbf{x}_1 + \mathbf{x}_2\right)/2}{\left|\mathbf{x}_0 - \left(\mathbf{x}_1 + \mathbf{x}_2\right)/2\right|}$$

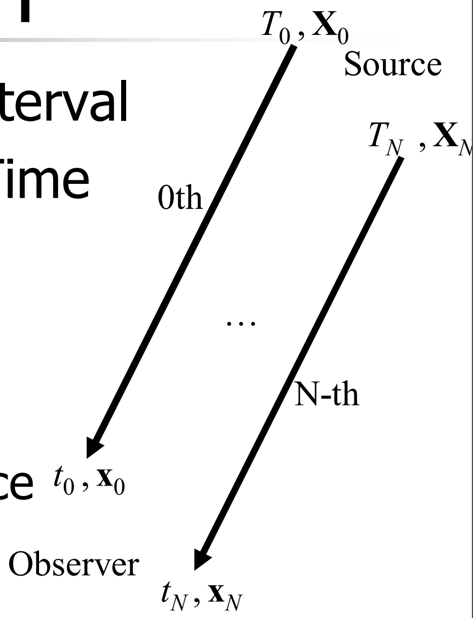
Quasi-Periodic Propagation

- Arrivals with Similar Interval
- Series of Eq. of Light Time
- Initial Arrival Epoch

$$\Delta t_N = t_N - t_0$$

- Assumption
  - Const. Interval at Source

$$\Delta T_N = T_N - T_0 = N \Delta T$$



Arrival-Time Observation Equation

- Diff. from Initial Eq. of Light Time
- Pulsar Arrival-Time Observation Eq.

$$c\Delta t_N = cN\Delta T - \mathbf{B}_N \cdot \mathbf{K}_0 + O\left(\frac{B_N}{R_0}\right)$$

- Baseline Vector ***B***
- Initial Direction ***K***

$$\mathbf{B}_N = (\mathbf{x}_N - \mathbf{x}_0) - (\mathbf{X}_N - \mathbf{X}_0)$$

$$\mathbf{K}_0 = \frac{\mathbf{X}_0 - \mathbf{x}_0}{|\mathbf{X}_0 - \mathbf{x}_0|}$$

11. Least Square Method (LSM)

- Gauss (1801): Ceres Orbit Determination
- Typical Optimization Problem
- Objective Function: $\Phi(\lambda)$

$$\Phi(\lambda) = \sum_j \left[f\left(t_j, \lambda\right) - g_j \right]^2$$

- Optimization = 0 PD of Objective Function = Set of Linear Equation (=Normal Eq.)

Application of LSM

- Data Analysis by Model Fitting
  - Linear Motion: Mean Place/Proper Motion
  - Kepler Ellipse: Binary Orbit Determination
  - Kepler Parabola: Comet Orbit Determination
  - Offset: Correction to Existing Model
  - Model Parameters: Geopotential Coefficients
  - Initial Conditions: Numerical Ephemeris
  - Proper Elements: Analytical Orbit Theory

Zero Partial Derivative

- Optimization = Zero PD $\frac{\partial \Phi}{\partial \lambda_i} = 0$
- Taylor Expansion

$$\left(\frac{\partial \Phi}{\partial \lambda_i}\right)(\lambda) = \left(\frac{\partial \Phi}{\partial \lambda_i}\right)_0 + \sum_j \left(\frac{\partial^2 \Phi}{\partial \lambda_i \partial \lambda_j}\right)_0 \Delta\lambda_j + \cdots$$

- Usage of Newton Method
- Normal Equation $\mathrm{H}\Delta\vec{\lambda} = -\vec{b}$

Normal Equation

- Hessian Matrix: Positive Def., Symmetric
- Standard: Modified Cholesky Method
- Caution!: Rank Deficiency, Degeneracy
- Recipe
  - General Inverse: Popular in Geodesy
  - Orthogonal Basis Expansion
  - Check Correlation Among Variables
  - Good Initial Guess

Extension of LSM

- Weighted LSM
  - Chi-Square Fitting
- Non-Linear LSM
  - Gaussian Approx., Quasi-Newton Method
- LSM Associated with Dynamical System
  - Integration of Variational Eq. of Motion

Error Estimation

- Variance-Covariance Matrix: Correlation among Parameters
- Diagonalization of Hessian Matrix
  - Determine Error Ellipsoid
  - Minimum of Obj. F.

$$\sigma_j = \sqrt{\frac{2\Phi_0}{H_{jj}}}$$

- No Meaning if Non-Diagonalized
- Practical Estimate: Very Difficult

12. Crush Course of General Rel. Effects

- Theories and Principles
- Galilean Approximation
- Newtonian Approximation
- Post-Galilean Approximation
- Post-Newtonian Approximation
- Dragging of Inertial Frame

Relativistic Theory

- Special Theory of Relativity
- Einstein's General Theory of Relativity
- Other Gravitational Theories
  - Brans-Dicke, Nordvegt, Ng, …
  - Scalar-Vector, Scalar-Tensor, …
  - Parametrized Post-Newtonian (PPN) Approximation

Principles

- Special Theory of Relativity (STR)
  - Principle of Special Relativity
  - Principle of Constancy of Light Speed
  - Principle of Limit of STR
- General Theory of Relativity (GTR)
  - Principle of General Relativity
  - Principle of Equivalence
  - Principle of Limit of GTR

Principle of Limit

- Unspoken but Important
- Special Theory of Relativity
  - Limit of Infinite c = Newton Mechanics
- General Theory of Relativity
  - Limit of Infinite c = Newton Mechanics + Law of Universal Attraction
  - Limit of Zero Gravity = STR

4-Dim. Spacetime

- 3+1 Dimension Expression

$$x^{\mu} \quad (\mu = 0,1,2,3) \qquad x^{0} = ct$$

- Metric Tensor

$$(\mathrm{d}s)^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}$$

Proper Time

- Definition $$c^2 \left(\mathrm{d}\tau\right)^2 = -\left(\mathrm{d}s\right)^2$$
- Reading of a Clock Moving with Observer
- 4-Velocity $$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

Galilean Metric

$$g_{\mu\nu} \cong \eta_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{G} \cong \underline{H} = \begin{pmatrix} -1 & \mathbf{0}^T \\ \mathbf{0} & \mathrm{I} \end{pmatrix}$$

Lorentz Transformation

- Basic Formula (1-D Space)

$$\begin{pmatrix} c\Delta\hat{t} \\ \Delta\hat{x} \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \end{pmatrix}$$

$$\psi = \tanh^{-1}\frac{v}{c}$$

- General Formula (3-D Space)

$$\underline{L} = \begin{pmatrix} \cosh\psi & (\sinh\psi)\ \mathbf{n}^T \\ (\sinh\psi)\ \mathbf{n} & (\cosh\psi)\ \mathbf{n}\otimes\mathbf{n} \end{pmatrix}$$

$$\mathbf{n} = \frac{\mathbf{v}}{v}$$

Poincare Transformation

- Natural Extension of Lorentz Transf.

$$x^{\hat{\alpha}}\left(x^{\mu}\right) = x^{\hat{\alpha}}{}_{O} + P^{\hat{\alpha}}_{\mu} x^{\mu}$$

- = Parallel Shift of Origin + Lorentz Transf. + Rotation

$$\underline{P} = \underline{L}\,\underline{R}$$

$$\underline{R} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{R} \end{pmatrix}$$

Newtonian Metric

- Gravitational Force Function ϕ
  - Note Signature: $\phi > 0$

$$\underline{G} \cong \begin{pmatrix} -1 + \dfrac{2\phi}{c^2} & \mathbf{0}^T \\ \mathbf{0} & \mathrm{I} \end{pmatrix}$$

Time Dilatation

- Newtonian Approximation

$$\frac{\mathrm{d}\tau}{\mathrm{dt}} \approx 1 - \frac{1}{c^2}\left(\phi + \frac{v^2}{2}\right) = 1 - \frac{\phi_{\mathrm{eff}}}{c^2}$$

- Lorentzian TD: Moving Clock Delays
- Gravitational TD: Delay Under Grav. F.
- Meaning of Effective Grav. Potential

Wavelength Shift

- Phase = Gauge Invariant
  - Independent on Choice of CS

$$\Delta\theta = 0 \Rightarrow \frac{\Delta\omega}{\omega} = \frac{\Delta f}{f} = \frac{-\Delta\tau}{\tau}$$

- 2^{nd} Order Doppler Shift
- Gravitational Red Shift

Post-Galilean Metric

$$\underline{G} \cong \begin{pmatrix} -1+\dfrac{2\phi}{c^2} & \mathbf{0}^T \\ \mathbf{0} & \left(1+\dfrac{2\gamma\phi}{c^2}\right)\mathrm{I} \end{pmatrix}$$

PPN Formalism

- C.F. Will (1981)
- Parametrized Post-Newtonian (PPN) F.
- PPN Parameters: α=1, β, γ, …
- α=1
  - Principle of Equivalence
  - One of Principles of Limit (GTR)

PPN Parameter

- GTR: $\beta = \gamma = 1$、他は0
- Nonlinearity of Grav. F.: β
- Spatial Curvature: γ
- All Experiments Support GTR
  - Planetary Motion: $\beta = 1.00$
  - Radio Bending by Sun: $\gamma = 1.000$

Geodesic

- Extension of Straight Line
  - Extended Law of Inertia in GTR
- Timelike Geodesic: World Line (WL) of Massive Particle
- Null Geodesic: WL of Particle with Zero Rest Mass (Photon, etc.)
- Spacelike Geodesic: Spatial Coord. Axis

Eq. of Geodesic

- Principle of Equivalence
  - Gravity is Not A Force
- Path of Free-Fall Particle = Geodesic
- Equation of Timelike Geodesic

$$a^{\mu} = \frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} + \Gamma^{\mu}_{\nu\rho} u^{\nu} u^{\rho} = 0$$

Christoffel's Symbol

- Not A Tensor = Depends on CS
  - Can Be 0 at One Point by Coord. Transf.
- Extension of Gravitational Acceleration

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\rho\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right)$$

- Inverse Metric Tensor $g^{\lambda\mu} g_{\mu\nu} = \delta^{\lambda}{}_{\nu}$

Eq. of Motion of Photon

- Path of Photon = Null Geodesic

$$\frac{dk^{\mu}}{d\lambda} + \Gamma^{\mu}_{\nu\rho} k^{\nu} k^{\rho} = 0$$

$$\Rightarrow \quad \frac{\mathrm{d}\mathbf{v}}{\mathrm{dt}} = \mathbf{0} + \left(\frac{1+\gamma}{c^2} \right) \left[\mathbf{a} - \frac{\left(\mathbf{a} \cdot \mathbf{v} \right) \mathbf{v}}{c^2} \right]$$

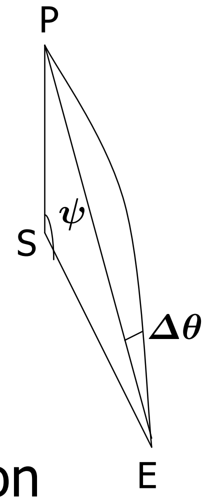
- Newtonian Gravitational Acceleration: **a**
- Solution by Successive Approximation

Gravitational Lensing

- Grav. Field = Convex Lense
- Deflection Angle

$$\Delta\theta = \frac{(1+\gamma)\mu_S}{c^2 r_{SE}} \tan\frac{\psi}{2}$$

- Large Defl.: 2~4 Images, Ring
- Microlensing = Light Amplification
  - Detection of MACHO

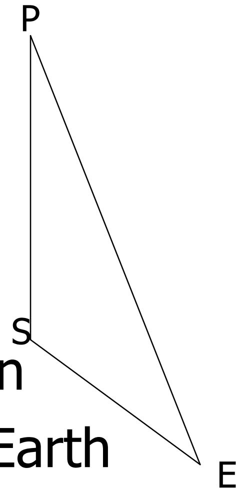


Gravitational Delay

- Shapiro Effect (Shapiro 1964)

$$\Delta\tau = \frac{(1+\gamma)\,\mu_S}{c^3}\log\left(\frac{r_{SE}+r_{SP}+r_{PE}}{r_{SE}+r_{SP}-r_{PE}}\right)$$

- Radar Bombing of Planets
- Pulsar Arrival Time Observation
  - Solar System: Sun, Jupiter, Earth
  - Binary Pulsar: Companion



Post-Newtonian Metric

- Nonlinear Scalar Potential $\Phi = \beta\phi^2 + \cdots$
- Vector (Gravito-Magnetic) Potential **g**

$$\underline{G} \cong \begin{pmatrix} -1 + \dfrac{2\phi}{c^2} + \dfrac{2\Phi}{c^4} & \dfrac{\mathbf{g}}{c^3}^{T} \\ \dfrac{\mathbf{g}}{c^3} & \left(1 + \dfrac{2\gamma\phi}{c^2}\right)\mathrm{I} \end{pmatrix}$$

4-Acceleration

- 4-Dim. Acceleration

$$a^{\mu} \equiv \frac{\mathrm{D}u^{\mu}}{\mathrm{d}\tau} = \frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} + \Gamma^{\mu}_{\nu\rho} u^{\nu} u^{\rho}$$

- Absolute Derivative, D
- Proper (=Rest) Mass, m
- 4-Force $f^{\mu} = m a^{\mu}$

PN Eq. of Motion

- EIH（Einstein, Infeld, Hoffmann）Eq. of Motion

$$\frac{\mathrm{d}\mathbf{v}_K}{\mathrm{dt}} = \mathbf{a}_K + \frac{1}{c^2}\sum_{J \neq K}\left(\frac{\mu_J}{r_{JK}}\right)\left[\frac{A_{JK}\mathbf{r}_{JK} + B_{JK}\mathbf{v}_{JK}}{r_{JK}^2} + (3+4\gamma)\mathbf{a}_J\right]$$

$$\mathbf{a}_K = \sum_{J \neq K}\frac{\mu_J \mathbf{r}_{JK}}{r_{JK}^3}, \quad \mathbf{r}_{JK} = \mathbf{r}_J - \mathbf{r}_K$$

PN Eq. of Motion (2)

$$\mathbf{v}_{JK} = \mathbf{v}_J - \mathbf{v}_K,$$

$$A_{JK} = -2(\beta+\gamma)\sum_{L\neq K}\frac{\mu_L}{r_{KL}} - (2\beta-1)\sum_{L\neq J}\frac{\mu_L}{r_{JL}} + \gamma\mathbf{v}_K^2$$

$$+(1+\gamma)\mathbf{v}_J^2 - 2(1+\gamma)\mathbf{v}_J\cdot\mathbf{v}_K - \frac{3}{2}\left(\frac{\mathbf{r}_{JK}\cdot\mathbf{v}_J}{r_{JK}}\right)^2 + \frac{\mathbf{r}_{JK}\cdot\mathbf{a}_J}{2},$$

$$B_{JK} = \mathbf{r}_{JK}\cdot\left[(2+2\gamma)\mathbf{v}_K - (1+2\gamma)\mathbf{v}_J\right]$$

Dragging of Inertial Frame

- Locally Parallel Shift of Origin $\neq$ Global Non-Rotation
  - No Coriolis Force $\neq$ Rest w.r.t. Quasar
- Fermi Transportation
  - GR Extension of Parallel Shift of Origin
- Proper CS = Fermi-Transported CS

Dragging of Inertial Frame (2)

- Rotation Velocity of Proper CS
- STR: Thomas Precession
- GTR
  - Geodesic Precession
    - ~1.92 arcsec/jc
    - De Sitter (1917)
  - Lense-Thirring Effect
    - Gravito-Magnetic Effect

$$\frac{\mathbf{v}\times\mathbf{a}}{c^3}$$

$$\frac{(1+\gamma)\,\mathbf{v}\times\nabla\phi}{c^3}$$

$$\frac{\nabla\times\mathbf{g}}{c^3}$$

12. References

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