THEORETICAL ASTROMETRY
Draft

Toshio Fukushima
National Astronomical Observatory
2-21-1, Ohsawa, Mitaka, Tokyo, 181-8588 Japan
Toshio.Fukushima@nao.ac.jp

March 19, 2003

Contents

Chapter 1

INTRODUCTION

1.1 What is Astrometry?

- Definition

  To study the nature of the universe by means of the measurements of positions and motions of celestial objects

- Place

  Fundamental Astronomy $\approx$ Classical Astronomy

  = Positional Astronomy $\approx$ Astrometry

  + Celestial Mechanics $\approx$ Dynamical Astronomy

  $\equiv$ Astronomy in the narrow sense $\approx$ Astronomy in *Astronomy and Astrophysics*

- Related Science

  Celestial Mechanics: motion of celestial objects

  Geodesy: Earth

  General Relativity: space and time

1.2 Text Books

Here are my personal notes of some textbooks on astrometry and related matters (in the chronological order of publishment). See references for the details.

- [Smart 1931]

  Classic but obsolete.

- [Woolard and Clemence 1966]

  Standard yet a little obsolete.

- [Taff 1981]

  Concise but too much computational.

- [Nagasawa 1981]

  Written in Japanese.

- [Murray 1983]

  Modern but having too much emphashis on general relativity. I don't think it suitable to begin by the geodesic equation in explaining the Keplerian motion.

- [Kovalevsky *et al.* (ed.) 1989]

  Comprehensive.

- [Soffel 1989]

  Suitable to understand Relativistic effects

- [Seidelmann (ed.) 1992]

  Detailed.

1.3 Units

The units used in the astrometry are classified into four categories; time, length, mass, and angle.

In general use of science and technology, it is recommended to follow completely the International System of Units, the *SI Units*[1] . However, some other traditional units, named as the astronomical units, have been used in the astronomy. Since they are useful, they will remain for the moment.

[1] The rigorous expression of *SI* is in French. Thus the order of its abbreviation is reverse to that in English. Similar things happen frequently in the terminology of time such as the Coordinated Universal Time *UTC* or the International Atomic Time *TAI*.

1.3.1 Time

The SI unit of time is *second*. It is defined by means of a certain frequency of Cesium atom (13th CGPM[2] , 1967) as

> *The second is the duration of 9162631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom.*

The symbol[3] is simply s. Do not use the old expression, sec. Auxiliary units commonly used are *minute, hour*, and *day*. Their symbols are $^{\mathrm{m}}$,$^{\mathrm{h}}$, and day. A typical usage is

$$t = 9\ \mathrm{day} 12^{\mathrm{h}} 34^{\mathrm{m}} 56^{\mathrm{s}}.7890.$$

Special auxiliary units, *Julian year* and *Julian century*, are used in astronomy. Their symbols are jy and jc, respectively. Sometimes the symbols yr and cy are used instead. A typical usage[4] is

$$\frac{\mathrm{d}\Delta\varpi}{\mathrm{dt}} = 43''.2\ \mathrm{jc}^{-1}.$$

The numerical relations among these units are

$$1\ \mathrm{day} = 24^{\mathrm{h}} = 1440^{\mathrm{m}} = 86400\ \mathrm{s}, \tag{1.1}$$

and

$$1\ \mathrm{jc} = 100\ \mathrm{jy} = 36525\ \mathrm{day} = 3.15576 \times 10^{9}\ \mathrm{s} \tag{1.2}$$

Remark that the amount of Julian year is somewhat larger than the so-called *mean solar year* T as

$$1\ \mathrm{jy} = 365.25\ \mathrm{day} > T \sim 365.2421897\ \mathrm{day} \tag{1.3}$$

where T is connected to P_{EMB}, the mean orbital period of the Earth-Moon barycenter, as

$$T = P_{\mathrm{EMB}} + 1\ \mathrm{day} \tag{1.4}$$

[2] CGPM stands for *Conférence Générale des Poids et Mesures*, the General Conference of Poids and Measures, which is one of inter-governmental committees.
[3] The expression of units are always in roman and not in *italic* nor in *slant*.
[4] This is the famous *advance of perihelion* of Mercury due to the general relativity

1.3.2 Length

The SI unit of length is *meter*. The symbol is simply m. It is derived from the SI unit of time and the speed of light in vacuum (17th CGPM, 1983) as

> *The meter is the length of the path travelled by light in vacuum during a time interval of 1/299792458 of a second.*

Namely the meter is mathematically defined such that the speed of light in vacuum, c, is expressed in the units, m and s, as

$$c = 299792458 \text{ ms}^{-1} \sim 3.0 \times 10^{8} \text{ ms}^{-1} \tag{1.5}$$

Thus, the number 299792458 has become one of golden numbers which scientists and engineers should remember; just like the first several digits[5] of π or the first several powers of 2.

The astronomical unit of length is AU, which is used to express the distances within the solar system, or those of similar lengths like the semi-major axis of an exoplanet[6] . Although its symbol is formally A, most people use simply AU instead. Sometimes instead of AU, we use the light time for the unit distance

$$\tau_A \equiv \frac{A}{c} \tag{1.7}$$

The numerical value of τ_A is

$$\tau_A \approx 499.00478353 \text{ s} \sim 500 \text{ s}. \tag{1.8}$$

The length of AU is quite close to the mean distance between the Sun and the Earth. It is roughly expressed in meter as

$$1 \text{ AU} \approx 1.495978706082166 \times 10^{11} \text{ m} \sim 1.5 \times 10^{11} \text{ m} \tag{1.9}$$

Another unit named *parsec* is used to express the distances to stars. Its symbol is pc. The length of parsec is defined as

$$1 \text{ pc} \equiv \frac{1}{\sin 1''} \text{ AU}$$

$$\approx 2.062648062470964 \times 10^{5} \text{ AU} \sim 2.1 \times 10^{5} \text{ AU}$$

[5] If arctangent function is accessible, sufficient number of digits of π are easily obtained as

$$\pi = 4 \tan^{-1} 1 \tag{1.6}$$

[6] The *exoplanet* means a planet moving around a star other than the Sun.

$$\approx 3.085677579610287 \times 10^{16} \text{ m} \sim 3.1 \times 10^{16} \text{ m} \tag{1.10}$$

In popular articles, the unit *light year* is frequently used. The symbol is ly. It is defined as the length which light runs during a year. If we regard the year as the Julian year, then

$$1 \text{ ly} \equiv c \times 1 \text{ jy} = 9.4607304725808 \times 10^{15} \text{ m} \sim 9.5 \times 10^{15} \text{ m} \tag{1.11}$$

Remark that the relation between the parsec and the light year is

$$1 \text{ pc} \approx 3.261563775179130 \text{ ly} \sim 3.26 \text{ ly} \tag{1.12}$$

1.3.3 Mass

The SI unit of mass is *kilogramm*. The symbol is kg. When using kg, however, The numerical values of masses of celetial bodies are quite large such as

$$M_{\text{Earth}} \approx 5.973699084395432 \times 10^{14} \text{ kg} \sim 6.0 \times 10^{14} \text{ kg} \tag{1.13}$$

Therefore, we need an auxiliary unit to express them compactly[7] .

The astronomical unit of mass is the mass of the Sun, M_{Sun}. A typical usage is

$$M_{\text{Ceres}} \approx 5.9 \times 10^{-9} M_{\text{Sun}}$$

The inverse ratios have been used in expressing the masses of major planets as

$$M_{\text{Jupiter}} \approx \frac{M_{\text{Sun}}}{1047.350}$$

Unfortunately, the numerical relation between kg and M_{Sun} is poorly determined as

$$M_{\text{Sun}} \approx 1.98892 \times 10^{20} \text{ kg} \sim 2.0 \times 10^{20} \text{ kg} \tag{1.14}$$

Why? The reason will be explained in the next subsection.

1.3.4 Gravitational Constant

Before explaining the relation between the astronomical unit of mass, M_{Sun}, and that of SI unit, kg, we give an important remark.

> *The masses of celestial objects are never directly determined by astronomical observations.*

[7] There is a more important reason to use the astronomical unit of mass. See the explanation in the next subsection.

In fact, the masses of celestial objects appear only in the form of *gravitational constant*, GM, namely the product of mass M and the Newton's universal constant of gravitation G. The unit of gravitational constant is that of the Sun, $GM_{\rm Sun}$, which is called as the *heliocentric gravitational constant.*

The gravitational constants, GM, can be precisely determined from the astrometric observations such as

$$GM_{\rm Earth} \approx 3.98600448 \times 10^{14} {\rm m}^3{\rm s}^{-2} \tag{1.15}$$

which is named as the *geocentric gravitational constant.* Since G is believed[8] to be constant anywhere in the universe at any time, the mass of an arbitrary celestial object is usually expressed in the ratio to that of the reference object such as the Sun or the Earth. In this sense, we may better rewrite the above relations of masses of planets as

$$GM_{\rm Ceres} \approx 5.9 \times 10^{-9}\ GM_{\rm Sun}, \qquad GM_{\rm Jupiter} \approx \frac{GM_{\rm Sun}}{1047.350}.$$

Remark that the dimension of GM is ${\rm L}^3\ {\rm T}^{-2}$. Thus the unit of GM is directly derived from those of length and time. For example, the heliocentric gravitational constant is a constant connected to AU and day as

$$GM_{\rm Sun} = k^2 {\rm AU}^3 {\rm day}^{-2} \tag{1.16}$$

where

$$k = 0.01720209895 \tag{1.17}$$

is a fixed number[9] named as the Gaussian gravitational constant[10] . As a result, the numerical value of the heliocentric gravitational constant in the SI units becomes

Remark that the numerical value of G itself is poorly determined as

$$G \approx 6.67259 \times 10^{-11}\ {\rm m}^3{\rm kg}^{-1}{\rm s}^{-2} \tag{1.18}$$

[8] Of course, this is not trivial fact. Many scientists including P.A.M. Dirac have worked to measure the change of G, especially its time variation. So far, only upper limits have been obtained as $\left|\dot{G}/G\right| < 10^{-20}$.

[9] The number cited here is exact.

[10] The Gaussian gravitational constant was originally defined as th daily mean motion, measured in radian, of a massless particle moving around the Sun with the semi-major axis being equal to 1 AU

1.3.5 Angle

The SI unit of angle is radian. The classic set of units of angle is *degree, arcminute*, and *arcsecond*[11] . Sometimes *revolution* is used as an auxiliary unit. These have been used in the astronomy.

The symbols are ° for degree, ' for arcminute, " for arcsecond, and $^{\rm rev}$ for revolution. A typical usage is

$$\theta = 9^{\rm rev}12^\circ 34' 56''.7890.$$

We call this the DMS expression[12] . The numerical relations among these units are

$$1^\circ = 60' = 3600'', \qquad 1^{\rm rev} = 2\pi \text{ radian} = 360^\circ. \tag{1.19}$$

Thus θ in the above example is translated as

$$\theta = 11709296''.7890 \approx 56.76827279479 \text{ radian}.$$

As for the numerical computation of angles, we recommend to use a factor K converting arcsecond to radian. Assume that an angle is given in the DMS expression. Then one may first convert it in arcseconds, next multiply K to it, and obtain the numerical value in radian.

The factor K is defined as the numerical value of arcsecond measured in radian as

$$K \equiv \frac{1''}{\text{radian}} = \frac{\pi}{180 \times 3600} \approx 4.848136811095360 \times 10^{-6} \sim 5 \times 10^{-6}. \tag{1.20}$$

It is better to remember this relation roughly as

$$20'' \sim 10^{-4} \text{ radian} \tag{1.21}$$

which is close to the magnitude of aberration. In other words, it is the magnitude of special relativistic effects around the Earth.

In expressing small angles, two new units have been commonly used nowadays. The one is milliarcsecond and the other is microarcsecond. Their symbols are *mas* and μ*as*. Here *as* is the abbreviation of arcsecond. Thus

$$1\text{mas} \equiv 10^{-3}\text{arcsecond} \sim 5 \times 10^{-9} \text{ radian} \tag{1.22}$$

and

$$1\mu\text{as} \equiv 10^{-6}\text{arcsecond} \sim 5 \times 10^{-12} \text{ radian} \tag{1.23}$$

[11] Sometimes, arcminute and arcsecond are called as *minute of arc* and *second of arc*, respectively. Although the words, arcminutes and arcseconds, are frequenty shortened as *minute* and *second*, we do not recommend this simplification because it is quite confusing with the units of time.

[12] DMS stands for Degree-(arc)Minute-(arc)Second.

Remark that, in the astronomy, some angles are expressed in the units of time. The examples are the right ascension of stars, α, and the angle of the Earth rotation, UT1. We call this the HMS expression[13] . A typical usage is

$$\alpha = 12^{\mathrm{h}}34^{\mathrm{m}}56^{\mathrm{s}}.7890.$$

The numerical relations between these angle-in-time units, ($^{\mathrm{h}}$,$^{\mathrm{m}}$,$^{\mathrm{s}}$), and the classical angle units, ($^\circ$,$'$,$''$), are

$$1^{\mathrm{h}} = 15^\circ, \qquad 1^{\mathrm{m}} = 15', \qquad 1^{\mathrm{s}} = 15''. \tag{1.24}$$

Thus the above example becomes

$$\alpha = 45296^{\mathrm{s}}.7890 \approx 0.2196050301753 \,\mathrm{radian}.$$

1.4 Quick Look of Astrometric Observation

- Classification:
  1. Passive
     Measurements of natural signals
     An example: photographs of images of stars
  2. Semi-passive
     Measurements of artificial signals
     An example: radio receiving of GPS signals
  3. Active
     Measurements of signals controled by observers themselves
     An example: lunar laser ranging

- Passive Observations
  1. Astrograph
  2. Meridian Circle/Transit
  3. Eclipse/Occultation Observation
  4. VLBI
  5. Optical/IR Interferometry
  6. Pulsar Timing Measurement

[13] HMS stands for Hour-Minute-Second.

- Semi-passive Observations

    1. Doppler observation of artificial objects
    2. GPS

- Active Observations

    1. Radio transpondering of artificial objects
    2. Laser ranging

Chapter 2

PRINCIPLES

2.1 Experienced Facts

Experiences show that the following rules[1] seem to govern the universe;

1. Four dimensional spacetime

   Our universe consists of one-dimensional time and three-dimensional space.

   We call a place in this four dimensional spacetime an *event*.

2. Continuity of spacetime

   The four dimensional spacetime is continuous.

   As a result, for any event P, the spacetime is separated into three sub-spacetimes depending whether its time is earlier than, the same as, and later than the time of P, respectively; the past of P, the present of P, and the future of$P$.

3. Rule of causality

   For any event, there exist at least a certain event which caused it.

   In general, the number of causing events is multiple. In an extreme case, the whole of the *past* sub-spacetime can be the causing events.

4. Definiteness of time arrow direction

   The causing event(s) happens in the past of the caused event.

5. Existence of inertial frame

[1] We ignore most of the quantum and relativistic effects in this treatment

There exists at least one frame of reference where the physical laws take the simplest form.

Actually, it is the frame where Newton's first law of motion (law of inertia) is valid. We name such frame the inertial frame.

6. Galilei's principle of relativity

   The physical phenomena are invariant with respect to the choice of reference frames as long as they are connected to the indertial frame by means of a linear transformation of time.

7. Principle of determinacy

   The state of universe is determined uniquely if their initial state is specified.

2.2 Coordinates

As we saw in Section 2.1, we need a set of four numbers in identifying a place in our universe. Such set of numbers are called as *coordinates*. There are two different type of coordinates; a coordinate specifying *when*, and three coordinates pointing *where*. We call the former the *time coordinate* and the latter the *spacial coordinates*. Frequently the former is abbreviated as *time*. While the latter is called as *position vector* or *position* simply.

Since time has only one dimension, there is no ambiguity in its mathematical expression. In fact, even if there exist multiple mathematical expressions of time, they are essentially the same since there should be one-to-one mapping between any pair of them. In this sense, the time is unique. In our treatment, we completely separate the issues on time from others. They will be given later in a separate Chapter.

On the other hand, there are variety of expressions of spatial coordinates. Some of them are fixed and some other are referred to moving reference frames. To give precise relations among them is one important task of the astrometry.

2.3 Essence of Astrometric Observation

The main physical phenomenon connected to the astrometric observation is a one-way propagation of signal in the universe. The largest difference between the propagation considered in the astrometry and other physical measurements is the distance dealt with is quite large, of the scale of the universe.

- Components

  1. Motion of Targets

2. Propagation of Signals
3. Motion of Observers
4. Projection to Reference Frames

- Typical Motions

1. Linear Motion
2. Rigid Body Rotation
3. Keplerian Motion
4. Perturbed Keplerian Motion

- Signal

1. ElectroMagnetic Wave/Photon
2. High Energy Particles

Chapter 3

TIME

Earlier, we claimed that the time has only one dimension. This means that its mathematical expression should be essentially unique. Assume that we have two different expressions of time. In that case, there should be a one-to-one mapping between them. Namely if we write the relation between these two times, t and τ for example, as

$$t = f(\tau), \tag{3.1}$$

then the function $f()$ must be a one-to-one mapping. Furthermore, the function must be monotonically increasing. Anyway, it is always possible to find such a mapping function. Thus, we will ignore the differences among the expressions and regard the time a unique thing hereafter.

3.1 Integrated Time Scale

Now, let us consider how to measure the time. Since we cannot travel from the future to the past, we can only once record the happening of an event. Thus, it is impossible to measure a time interval repeatedly. Namely, the repeatablity of measurements, which is the key feature of modern metrology, is not applicable to the time itself, which is measurable most precisely[1] . Then, one of the rest possibilities is

1. to assume all the time intervals of a certain repeating phenomenon the same,
2. to count the number of its repetitions (and its phase if possible), and
3. to regard the counted number (plus the phase) as a representation of time.

The time realized by this procedure is called as the *integrated time scale.*

[1] This is quite an ironical situation

3.2 Atomic Time

A typical case of the integrated time scale is the atomic time, which is maintained by atomic clocks. An atomic clock represents an integrated number of a certain oscillation of a certain atom like cesium. See Section 1.3.1 for example.

3.3 International Atomic Time

To serve as the international standard of atomic times, the International Bureau of Poids and Mesures (BIPM) is currently compiling an ensemble average of atomic times. It is named as the *International Atomic Time* (TAI) and is disseminated worldwide. The formal definition of TAI (14th CGPM, 1971) is quite practical as;

> *International Atomic Time (TAI) is the time reference coordinate established by the Bureau International de l'Heure (BIH)*[2] *on the basis of the readings of atomic clocks operating in various establishements in accordance with the definition of the second, the unit of time of the International System of Units.*

An academic definition (9th CCDS, 1980) was given by *Comité Consultantif pour la Définition dela Seconde* (CCDS), one of the committees under CGPM, as

> *TAI is a coordinate time scale defined in a geocentric reference frame with the SI second as realized on the rotating geoid as the scale unit*

Here the geoid is the equipotential surface of the gravity of the rotating Earth, which includes not only the Newton's gravitational potential but also the centrifugal potential caused by the Earth rotation.

3.4 Universal Time

Another integrated time, which had been widely used in the past, is Universal Time (UT). The UT was defined as the integrated time by counting the number of Earth rotations and measuring their phase. Thus the UT would be uniform if the Earth rotation is uniform as the atomic frequency standard.

If the reference point, namely the direction in space which the Earth rotation is counted, is the Sun, the UT was called as the solar time. Since the motion of the Sun in space seen from the Earth is not uniform[3] old astronomers introduced an artifact object named as

[2] In 1985, the time section of BIH was moved to BIPM.

[3] First of all, the approximate Keplerian orbit has a small but finite eccentricity. Next there come perturbations of the Moon and other planets.

the *fictitious mean Sun*, which runs with a constant angular velocity in space while keeping the same orbital period as that of the actual Sun. The solar time referred to this fictitious mean Sun was called as the mean solar time. Its time unit was the mean solar day, which is the period from the time when the mean Sun passes the Greenwich meridian to the time when it passes the meridian once more.

3.5 Coordinated Universal Time

The current civil time is based on the Coordinated Universal Time, UTC. The UTC is defined[4] as a time scale offset from the TAI by a certain integral TAI seconds such that its difference from UT1 be within 1 second. More precisely,

$$\mathrm{UTC} = \mathrm{TAI} + n\mathrm{s}_{\mathrm{TAI}} \tag{3.2}$$

where the integer n is adjusted such that

$$|\mathrm{UTC} - \mathrm{UT1}| < 1\mathrm{s} \tag{3.3}$$

The adjustment, i.e. the increase (or decrease) of the difference UTC−TAI, is conducted by a joint service of the IAU and IAG named the International Earth Rotation Service (IERS), which is in charge of the monitoring the Earth rotation (i.e. UT1). The insertion is done at the end of half year, namely after the last second of June 30th or of December 31st. This additional second is called the leap second. The insertion goes like this;

$$\cdots, 23^{\mathrm{h}}59^{\mathrm{m}}58^{\mathrm{s}},\ 23^{\mathrm{h}}59^{\mathrm{m}}59^{\mathrm{s}},\ 23^{\mathrm{h}}59^{\mathrm{m}}60^{\mathrm{s}},\ 0^{\mathrm{h}}00^{\mathrm{m}}00^{\mathrm{s}},\ 0^{\mathrm{h}}00^{\mathrm{m}}01^{\mathrm{s}},\ \cdots$$

In the above, the leap second is tagged as $23^{\mathrm{h}}59^{\mathrm{m}}60^{\mathrm{s}}$.

3.6 Pulsar Time

3.7 Dynamical Time Scale

Another time scale being conceptually important is dynamical time scale. Assume a certain astronomical phenomenon is expressed as a definite one-to-one mapping function of time as

$$p = P(t) \tag{3.4}$$

[4] The current definition has been effective since April 1977. Before that, the conversion between UTC and TAI is very complicated. Refer Chapter 2 of Seidelmann et al. (1992).

where p denotes the status of the phenomenon such as the phase angle of Moon for example. Then, the time scale defined by the inverse function

$$t \equiv P^{-1}(p) \tag{3.5}$$

is called as the dynamical time.

An example of such phenomenon is the mean longitude of the Sun, L. In the past, just after the nonuniformity of the UT was revealed, a time scale was set such that L is expressed by the following quadratic polynomial of time T;

$$L = 279^\circ 41' 48''.04 + 129602769''.13T + 1''.089T^2 \tag{3.6}$$

Here T is measured in the tropical year, which is defined as a period containing 31556925.97474 seconds, since B1900.0, the begining of Besselian year 1900. This was called *Ephemeris Time* (ET).

3.8 Barycentric Dynamical Time

During the years 1984 to 1991, the IAU adopted two astronomical time scales, TDB and TDT, in place of the previous single astronomical time scale; ET. They are the abbreviation of the solar-system-Barycentric Dynamical TIme (TDB), and the Terrestrial Dynamical Time (TDT). The time scale TDT is defined such that

$$\mathrm{TDT} = \mathrm{TAI} + 32.184\mathrm{s} \tag{3.7}$$

The constant offset was introduced to assure the continuation from ET to TDT at the epoch of transition, 1st January 1984. While, the TDB was defined to represent a time scale specifi to the solar system within the general relativistic framework. In the relativistic theories, such time scale naturally has a rate being different from that of TDT, which suffers the relativistic effect due to the gravitational potential of the Sun and planets as well that of the geoid. This difference in rate was thought to be inconvenient at the time of introduction of general relativistic concepts into the IAU. Thus the TDB was defined such that its rate is the same as that of the TDT as

$$< \mathrm{TDB} > = < \mathrm{TDT} > \tag{3.8}$$

Remark that this does not mean TDB=TDT. Actually there can be periodic differences.

3.9 Barycentric Coordinate Time

Since 1991, the IAU has introduced the current set of astronomical time scales; TCB, TCG, and TT, They are the abbreviations of the *solar-system-Barycentric Coordinate Time* (TCB), the *Geocentric Coordinate Time* (TCG), and the *Terrestrial Time* (TT), respectively.

3.10 Broadcast Time

3.11 Standard Times

The current civil times are connected to UTC with a constant amount of time difference. We call them the standard times. Examples are the EST (U.S. East Coast Standard Time) or JST (Japan Standard Time). Usually the time difference of standard times from UTC are set as integer multiples of hours and/or half hours. This is for ease of conversion. For example, the Japanese Standard Time is defined as

$$\mathrm{JST} = \mathrm{UTC} + 9^{\mathrm{h}} \tag{3.9}$$

Take much care with the meaning of this equation. This means that *the reading of JST is that of UTC plus 9 hours.* Namely, the instant when $\mathrm{UTC} = 0^{\mathrm{h}}$ is the same as the instant when $\mathrm{JST} = 9^{\mathrm{h}}$ of the same day.

In recording the astronomical observations, one must use UTC, which are transformed from the standard time of observatory by a similar transformation like the above.

3.12 Julian Date

From an academic point of view, the expression of time has been confusing and problematic. Even for the dates, i.e. the order of day, month, and year, there are a variety of ways. The standard one in English is, 'Month Day, Year', like September 12, 1998. Another is, 'Day Month Year', like 9 December 1998. A common problem occurs in expressing the date by numerics. For example, the above two *different* dates may be expressed in the same form as 09/12/98. Also, there is another ambiguity of the expression of year in its last two digits[5]

However, the most inconvenience appears when one wants to calculate the time intervals between two dates of different years. Can you compute the days from December 1st, 1876 to May 3rd, 1991 at once? To do that easily, astronomers have introduced a convenient system; Julian date. The Julian date (JD) is an integrated number of days from a very old date such that most of timings of astronomical observations are expressed by positive numbers of JD. The epoch of JD is the noon at Greenwich (i.e. at the prime meridian) of the date, 1 January 4713 B.C. The JD of the most useful epoch, J2000.0, is defined as

$$\mathrm{JD}_{\mathrm{J2000.0}} = 2451545.0 \tag{3.10}$$

which corresponds to the noon at Greenwich of the date, 1 January 2000. Remark that the JD when $\mathrm{UTC} = 0^{\mathrm{h}}$ has a fraction, 0.5.

[5] This is the so-called *Year 2000 Problem.*

In order to concide the beginning of UTC day and that of Julian-like dates, a modified date named MJD, Modified Julian Dates, has been introduced. The definition of MJD is

$$\mathrm{MJD} = \mathrm{JD} - 2400000.5 \tag{3.11}$$

As an example,

$$\mathrm{MJD}_{\mathrm{J2000.0}} = 51544.5 \tag{3.12}$$

3.13 Conversion to/from Julian Date

Let us express a timing in the calendar dates as (Y, M, D, H, m, S) where Y denotes the integer representing the year in A.D., M does the integer doing the month in the year, D does the integer date in the month, H does the integer doing the hour, m does the integer doing the minute, and S does the floating point number doing the second with its fraction. The conversion between it and the corresponding JD is done by using the following procedure developed by Fliegel and Van Flandern (1968);

$$\begin{aligned}
&\mathrm{JD2} = H/24.0 + m/1440.0 + S/86400.0, \qquad L = \mathrm{int}((M-14)/12), \\
&I = 1461 * (Y + 4800 + L), \qquad J = 367 * (M - 2 - 12 * L), \\
&K = \mathrm{int}((Y + 4900 + L)/100), \\
&\mathrm{JD0} = \mathrm{int}(I/4) + \mathrm{int}(J/12) - \mathrm{int}((3 * K)/4) + D - 32075, \\
&\mathrm{JD1} = \mathrm{JD0} + 0.5, \qquad \mathrm{JD} = \mathrm{JD1} + \mathrm{JD2}
\end{aligned} \tag{3.13}$$

where the function int means the operator to truncate off if not integral.

We remark that to express JD by a single double precision variable often causes the loss of precision even if the above conversion procedure is carefully coded. Thus, we recommend to express JD by a pair of double precision variables; (JD1, JD2) in the above notation.

If so, the procedure of backward conversion is;

$$\begin{aligned}
&\mathrm{JD0} = \mathrm{int}(\mathrm{JD} - 0.5), \qquad \mathrm{JD1} = \mathrm{JD0} + 0.5, \qquad \mathrm{JD2} = \mathrm{JD} - \mathrm{JD1} \\
&L = \mathrm{JD0} + 68569, \qquad N = \mathrm{int}((4 * L)/146097), \qquad K = L - \mathrm{int}((146097 * N + 3)/4), \\
&I = \mathrm{int}((4000 * (K + 1))/1461001), \qquad P = K - \mathrm{int}((1461 * I)/4) + 31 \\
&J = \mathrm{int}((80 * P)/2447), \qquad D = P - \mathrm{int}((2447 * J)/80), \qquad Q = \mathrm{int}(J/11), \\
&M = J + 2 - 12 * Q, \qquad Y = 100 * (N - 49) + I + Q
\end{aligned} \tag{3.14}$$

These two procedure is effective to all cases where JD is nonnegative.

Finally, we present a formula providing the day of week from JD;

$$I = JD - 7 * \mathrm{int}((JD + 1)/7) + 2 \tag{3.15}$$

where I runs from 1 to 7 while $I = 1$ denotes Sunday.

Chapter 4

POSITION AND VELOCITY

4.1 Observables and Coordinates

In expressing positions, we use an arbitrary chosen set of three numbers named as spatial coordinates. The typical example is three dimensional cartesian coordinates as

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{4.1}$$

The corresponding components of the velocity is defined as the time variation of the coordinates as

$$\vec{v} \equiv \frac{\mathrm{d}\vec{r}}{\mathrm{dt}} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}. \tag{4.2}$$

However, remark that neither coordinates nor their velocity components are the astrometric observables. As we saw in Section 1.4 quickly, the actual observables in the Newtonian framework[1] are

1. the time of an event[2] ,

2. the relative distance between a pair of positions[3] ,

[1] In the relativistic framework, all these reduce to the primary one; the time measurements.

[2] A typical example is the arrival of light signal as radio pulses from a pulsar.

[3] The possibility of the measurement of relative length relies on the availability of a rigid body. named as the scale of length, The existence of a rigid body is allowed in the Newtonian framework but not in the relativistic framework.

3. the relative angle between a pair of two directions[4] , and

4. the time variations of above observables

If we construct coordinate systems from the viewpoint of observer only, it would be quite different from ordinary cartesian ones. As an illustration to show a promising form of coordinates, we will do such a simulation in the next subsection.

4.2 Local Coordinate Systems

Let us put the discussion about time aside for the moment, and concentrate ourselves with the position. Then, from the viewpoint of an observer, the most convenient set of spatial coordinates would be such that

1. the coordinate origin is the observer's location, and

2. the coordinates are constructed from the *local observables*, namely the quantities which the observer can easily measure by himself/herself.

It is clear that one candidate of such coordinates is the distance from the coordinate origin

$$r = |\vec{r}| \tag{4.3}$$

Since the dimension of spatial coordinates is three, we need two more. The major possibilities[5] are the angles measured from two independent directions;

$$\theta_j = \cos^{-1} \frac{\vec{r} \cdot \vec{d}_j}{r}, \qquad (j = 1, 2) \tag{4.4}$$

where $\vec{d}_j$ denotes unit vectors respresenting such directions.

These directions $\vec{d}_j$ must be easily accessable to the observer and preferrably defined in a clear way. For the observers on the Earth, one candidate of such directions is the plumb line, $\vec{z}$, namely the local direction of gravitation. This is easily realized[6] by pending a thread with a weight at the bottom, or, being equivalent, by preparing a plane perpendicular to the gravitational direction with the help of a water level. Another candidate is the north or south pole, $\vec{p}$, i.e. the axis of Earth rotation. It is roughly realized by the direction of

[4] The possibility of the angle length also relies on the availability of a rigid body named as the circle scale of angle,

[5] In principle, additional distances from other one or two fixed points can be used instead. However, they are not locally availlable.

[6] We do not discuss the precision for the moment

polar star in the northern hemisphere or by a magnetic compass. Except for the observers located on the poles, these two directions are independent with each other as

$$\vec{z} \cdot \vec{p} = \sin\phi \tag{4.5}$$

where ϕ is the latitude of the observer's location. Thus one plane is defined so as to contain both directions. The plane is called as the *meridian*.

The transformation formulas from such locally defined coordinates to the rectangular coordinates would be easier if the two directions adopted are orthogonal to each other. Unfortunately, $\vec{z}$ is not perpendiculr with $\vec{p}$ except when the observer locates on the equator. Thus, there come two different systems of coordinates; the one prefers $\vec{z}$, and the other keeps $\vec{p}$.

The former system adopts $\vec{z}$ as the z-axis. The x-y plane is defined as the plane passing the coordinate origin and being orthogonal to the plumb line. It is called as the *horizon* or the horizontal plane. The x-axis is taken as the intersection of the meridian and the horizontal plane. Thus constructed rectangular coordinates is called as the horizontal coordinates.

While, the other way is called as the equatorial coordinates. There, the z-axis is set as the pole direction $\vec{p}$. And the x-y plane is defined as a plane which passes the coordinate origin, the observer, and is perpendicular to the pole direction. The plane is called as the *equator* or the equatorial plane. The x-axis is taken as the intersection of the meridian and the equator.

Anyway the promising coordinates are the spherical coordinates centered at the observer. Thus, we will give a general discussion on the spherical coordinates in the next subsection.

4.3 Spherical Coordinates

In astrometry, position vectors are frequently expressed in the spherical coordinates, i.e. three dimensional polar coordinates, as

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \sin\theta\cos\lambda \\ \sin\theta\sin\lambda \\ \cos\theta \end{pmatrix} = r \begin{pmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{pmatrix} \tag{4.6}$$

where r is the radius vector, θ is the zenith angle, ϕ is the latitude[7] and λ is the longitude. Their domains are

$$r \geq 0, \qquad |\theta| \leq \frac{\pi}{2}, \qquad |\phi| \leq \frac{\pi}{2}, \qquad 0 \leq \lambda < 2\pi \tag{4.7}$$

[7] There is another kind of latitude named as *spheroidal latitude*, φ. In comparison with the spherical latitude, this latitude is called as the *spherical* latitude.

Remark that the convention of trigonometric functions for the former pair (θ, λ) is named as the physical convention. While that for the latter pair (ϕ, λ) is called as the geographical convention, which is commonly used in astronomy.

A typical example is the expression in the *equatorial coordinate system*;

$$\vec{r} = r \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix} \tag{4.8}$$

where α does the *right ascension* and δ denotes the *declination.*

Another is the expression in the *ecliptic coordinate system*;

$$\vec{r} = r \begin{pmatrix} \cos\beta\cos\lambda \\ \cos\beta\sin\lambda \\ \sin\beta \end{pmatrix} \tag{4.9}$$

where β denotes the *ecliptic latitude* and and λ does the *ecliptic longitude.*

Yet another is the expression in the *horizontal coordinate system*;

$$\vec{r} = r \begin{pmatrix} \cos a\cos A \\ -\cos a\sin A \\ \sin a \end{pmatrix} \tag{4.10}$$

where a denotes the *altitude angle* and and A does the *azimuthal angle.* Remark that the angle A is measured *clockwise.*

The inverse relation between the sperical coordinates and the rectangular ones is well known as

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{4.11}$$

and

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \mathrm{atan2}\left(\sqrt{x^2 + y^2}, z\right), \tag{4.12}$$

$$\phi = \sin^{-1}\left(\frac{z}{r}\right) = \mathrm{atan2}\left(z, \sqrt{x^2 + y^2}\right), \tag{4.13}$$

$$\lambda = \mathrm{atan2}(y, x) \tag{4.14}$$

where $\mathrm{atan2}(y, x)$ is the function providing $\tan^{-1}(y/x)$ while taking the signatures of x and y into account such as implemented in Fortran function libraries. Remark that the domain of $\mathrm{atan2}(y, x)$ is in the range

$$|\mathrm{atan2}(y, x)| \le \pi \tag{4.15}$$

Therefore, in order to obtain nonnegative angle like α or λ, one should add 2π when the output is negative.

An example of transformation between spherical coordinates and rectangular coordinates is given in Tables 4.1 and 4.2.

Table 4.1: Example Computation of Rectangular Coordinates from Spherical Coordinates

Formula	Numerical Example
Spherical Coordinates	
r	10 AU
α	$12^{\mathrm{h}}34^{\mathrm{m}}56^{\mathrm{s}}.7890123$
δ	$+22^{\circ}59'14''.864151$
Conversion I	
α	$45296^{\mathrm{s}}.7890123$
δ	$82754''.864151$
Conversion II	
α	3.294075453524272 radian
δ	0.4012069031876588 radian
Computation	
$\sin\alpha$	-0.1518925901288032
$\cos\alpha$	-0.9883970057947178
$\sin\delta$	$+0.3905296898721468$
$\cos\delta$	$+0.9205903330626305$
Rectangular Coordinates	
$x = r\cos\delta\cos\alpha$	-9.099087287626659 AU
$y = r\cos\delta\sin\alpha$	-1.398308501364206 AU
$z = r\sin\delta$	$+3.905296898721468$ AU

Table 4.2: Example Computation of Spherical Coordinates from Rectangular Coordinates

Formula	Numerical Example
Rectangular Coordinates	
x	-9.099087287626659 AU
y	-1.398308501364206 AU
z	$+3.905296898721468$ AU
Computation	
$r = \sqrt{x^2 + y^2 + z^2}$	10.00000000000000 AU
$\alpha_0 = \mathrm{atan2}(y, x)$	-2.989109853655315 radian
$\rho = \sqrt{x^2 + y^2}$	9.205903330626304 AU
$\delta = \mathrm{atan2}(z, \rho)$	0.4012069031876589 radian
$\delta' = \sin^{-1}(z/r)$	0.4012069031876588 radian
Conversion I	
$\alpha = \alpha_0 + 2\pi$	3.294075453524272 radian
Conversion II	
α	$45296^{\mathrm{s}}.7890123000$
δ	$82754'.86415100000$
Spherical Coordinates	
r	10.00000000000000 AU
α	$12^{\mathrm{h}}34^{\mathrm{m}}56^{\mathrm{s}}.7890123000$
δ	$+22^\circ 59' 14''.86415100000$

Note: We adopted the expression $\mathrm{atan2}(z, \rho)$ in the actual computation of δ later on.

4.4 Velocity in Spherical Coordinates

The velocity is expressed in the spherical coordinates[8] as

$$\vec{v} = v_r \vec{e}_r + v_\phi \vec{e}_\phi + v_\lambda \vec{e}_\lambda \tag{4.16}$$

where

$$v_r = \frac{\mathrm{d}r}{\mathrm{dt}}, \qquad v_\phi = r\frac{\mathrm{d}\phi}{\mathrm{dt}}, \qquad v_\lambda = r\cos\phi\frac{\mathrm{d}\lambda}{\mathrm{dt}} \tag{4.17}$$

and

$$\vec{e}_r \equiv \left(\partial \vec{r} r\right)_{\phi,\lambda} = \begin{pmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{pmatrix}, \tag{4.18}$$

$$\vec{e}_\phi \equiv \frac{1}{r}\left(\partial \vec{r} \phi\right)_{r,\lambda} = \begin{pmatrix} -\sin\phi\cos\lambda \\ -\sin\phi\sin\lambda \\ \cos\phi \end{pmatrix}, \tag{4.19}$$

$$\vec{e}_\lambda \equiv \frac{1}{r\cos\phi}\left(\partial \vec{r} \lambda\right)_{r,\phi} = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix} \tag{4.20}$$

are the coordinate triad representing the horizontal reference frame at the position (r, ϕ, λ).

4.5 Spheroidal Coordinates

Sometimes, especially in describing the location on the Earth and other planets, position vectors are expressed in the spheroidal[9] coordinates (φ, λ, h). The variables are the spheroidal latitude[10] , φ, the longitude, λ, and the height from the reference spheroid[11] , h. They are connected to the cartesian coordinates (x, y, z) as

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho_N \cos\varphi\cos\lambda \\ \rho_N \cos\varphi\sin\lambda \\ \rho_Z \sin\varphi \end{pmatrix} \tag{4.21}$$

[8] We give here the geometrical convention only

[9] They are sometimes called as *ellipsoidal* coordinates. We will use the word *spheroidal* throughout this treatment since we want to reserve the word *ellipsoidal* for denoting the characters for ellipsods of three axes with different lengths.

[10] The spheroidal latitude is defined as the angle between the normal of the meridian ellipse and the equatorial plane. The quantity φ is called as the *geographic* latitude in the case of positions on the Earth. In that case, the spherical latitude ϕ is called as the *geocentric* latitude.

[11] Sometimes the word *reference ellipsoid* is used instead.

where

$$\rho_N = N(\varphi) + h, \qquad \rho_Z = \left(1 - e^2\right) N(\varphi) + h, \tag{4.22}$$

and

$$N(\varphi) \equiv \frac{a}{d(e, \varphi)} \tag{4.23}$$

is the (spheroidal) radius of curvature across the meridian[12] and

$$d(e, \varphi) \equiv \sqrt{1 - e^2 \sin^2 \varphi} \tag{4.24}$$

is the normalized radius[13] at the latitude φ. Here a and e are the equatorial radius[14] and the eccentricity of the spheroid. Namely the spheroid is defined by the equation

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{4.25}$$

where

$$b = ae' \tag{4.26}$$

is the polar radius[15] and

$$e' \equiv \sqrt{1 - e^2} = 1 - f \tag{4.27}$$

is the complementary eccentricity[16] . Remark that

$$d(0) = 1, \qquad d\left(\frac{\pi}{2}\right) = e'. \tag{4.28}$$

Not the eccentricity

$$e \equiv \sqrt{\frac{a^2 - b^2}{a^2}} \tag{4.29}$$

---

[12] Originally, $N(\varphi)$ was defined as the length of normal of the meridian ellipse at the location specified by the spheroidal latitude, φ. The symbol N stands for the *normal*. Mathematically, $N(\varphi)$ is the radius of curvature of the cross section of the spheroid cut across the meridian. In the terminology of differential geometry, $N(\varphi)$ is the second radius of curvature, or the radius of torsion.

[13] The trio $(\sin\varphi, \cos\varphi, d(e, \varphi))$ appears frewuently in the mathematical treatments of ellipse. The typical example is the trio of Jacobi's elliptic functions, $(\mathrm{sn}(u), \mathrm{cn}(u), \mathrm{dn}(u))$ where the argunment u is connectes to the angle φ by the amplitude function $\varphi = am(e, u)$. Or u is expressed by the first incomplete elliptic integral as $u = F(e, \varphi)$.

[14] Sometimes the symbol a_e is used instead.

[15] Sometimes the symbol a_p is used instead.

[16] Remark that e' is best derived not from e but from f.

but the flattening factor,

$$f \equiv \frac{a-b}{a} \tag{4.30}$$

is usually used in specifying the reference spheroid of the Earth. The relation[17] between e and f is

$$e^2 = 2f - f^2 \tag{4.31}$$

An example computation of the spheroidal coordinates is given in Table 4.3. Remark that the spheroidal latitude φ differs from the spherical latitude ϕ as

$$\varphi - \phi \sim \frac{e^2}{2} \sin 2\varphi \tag{4.32}$$

whose maximum is around $10'$ for the case of the Earth.

The inverse procedure, namely to obtain (φ, λ, h) from (x, y, z) given, is not straightforward as was in the spherical coordinates. Its details will be given in Appendix.

4.6 Velocity in Spheroidal Coordinates

The velocity is expressed in the spheroidal coordinates as

$$\vec{v} = v_h \vec{e}_h + v_\varphi \vec{e}_\varphi + v_\lambda \vec{e}_\lambda \tag{4.33}$$

where

$$v_h = \frac{\mathrm{d}h}{\mathrm{dt}}, \qquad v_\varphi = \rho_M \frac{\mathrm{d}\varphi}{\mathrm{dt}}, \qquad v_\lambda = \rho_N \cos\varphi \frac{\mathrm{d}\lambda}{\mathrm{dt}}. \tag{4.34}$$

Here

$$\rho_M = M(\varphi) + h \tag{4.35}$$

and

$$M(\varphi) = \frac{a\left(1-e^2\right)}{\left(\sqrt{1-e^2\sin^2\varphi}\right)^3} \tag{4.36}$$

[17] Remark that not e but e^2 only is required in the actual computation. Therefore, we recommend to keep the value of e^2 and never to take its square root.

Table 4.3: Example Computation of Spheroidal Coordinates

Formula	Numerical Example
Parameters	
a	6378136m
$1/f$	298.257
Preparation I	
f	0.003352813177896914
$e^2 = 2f - f^2$	0.006694384999587949
Geodetic Coordinates	
φ	45°
λ	30°
h	1000m
Preparation II	
$N(\varphi) = a/\sqrt{1 - e^2 \sin^2 \varphi}$	6388837.296471346
$\rho_N = N(\varphi) + h$	6389837.296471346
$\rho_Z = (1 - e^2)\, N(\varphi) + h$	6347067.959909041
Rectangular Coordinates	
$x = \rho_N \cos\varphi \cos\lambda$	3912960.228939990
$y = \rho_N \cos\varphi \sin\lambda$	2259148.641506802
$z = \rho_Z \sin\varphi$	4488054.795103548

is the (spheroidal) radius of curvature in the meridian[18] and

$$\vec{e}_h \equiv (\partial \vec{r} h)_{\varphi,\lambda} = \begin{pmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{pmatrix}, \tag{4.37}$$

$$\vec{e}_\varphi \equiv \frac{1}{\rho_M} (\partial \vec{r} \varphi)_{h,\lambda} = \begin{pmatrix} -\sin\varphi\cos\lambda \\ -\sin\varphi\sin\lambda \\ \cos\varphi \end{pmatrix}, \tag{4.38}$$

$$\vec{e}_\lambda \equiv \frac{1}{\rho_N \cos\phi} (\partial \vec{r} \lambda)_{h,\varphi} = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix} \tag{4.39}$$

are the coordinate triad representing a local horizontal reference frame at the position (φ, λ, h). Here we used the relations

$$\frac{\mathrm{d}\left(\rho_N \cos\varphi\right)}{\mathrm{d}\varphi} = \rho_M \sin\varphi, \qquad \frac{\mathrm{d}\left(\rho_Z \sin\varphi\right)}{\mathrm{d}\varphi} = \rho_M \cos\varphi, \tag{4.40}$$

which was derived from

$$N'(\varphi) = \frac{a e^2 \sin\varphi\cos\varphi}{\left(\sqrt{1 - e^2 \sin^2\varphi}\right)^3} \tag{4.41}$$

4.7 Reference Frames

As we saw in Sect.2.1, the time coordinate is dealt in a completely separate manner from three spatial coordinates in the Newtonian mechanics, on which our treatise of astrometry is based. Therefore, we will deal with the transformation among various coordinate systems in three spatial dimensions. In that case, we will use the word *reference frame* instead of three dimensional spatial coordinate system[19]

There are two ways in specifying the reference frame; conceptual and relational. The conceptual way is to give the definitions of three major elements of the reference frame; the coordinate origin, and the directions of the x- and z-axes. Sometimes, the x-y plane is

[18] Mathematically, $M(\varphi)$ is the radius of curvature of the meridian ellipse. The symbol M stands for *meridian*. In the terminology of differential geometry, $M(\varphi)$ is the first radius of curvature.

[19] Originally, the word *reference frame* means *coordinate triad*, namely a trio of orthonormal vectors. In rectangular coordinate systems, one may consider that a coordinate triad is equivalent with a coordinate system. This is not adequate in curvilinear coordinate systems, which inevitablly appear in general relativistic treatments.

selected instead of the z-axis, although the direction of z-axis becomes ambiguous in this case.

An example is the equatorial reference frame, which is defined as the reference frame satisfying the following conditions;

- the coordinate origin is the geocenter, the solar-system-barycenter, or the Sun,
- the x-axis is the vernal equinox, and
- the x-y plane is the equator.

Sometimes adjectives describing the above items more specifically are attached as *mean geocentric equatorial reference frame of date*, which means that

1. the coordinate origin is the geocenter,
2. the equinox and the equator are of *mean* nature, namely they contain the effect of precession only,
3. the equinox and the equator are those of present time, t.

On the other hand, the relational way is to give a form of coordinate transformation from the existing reference frame. This is more rigorous than the former way. Although there remain a fundamental problem to define one reference frame at least without the relational ways of defining.

4.7.1 Transformation of Coordinate Systems

The general form of three dimensional coordinate transformation from the old to the new coordinate systems is expressed as

$$\vec{r}_P^{\,(\mathrm{NEW})} = \vec{r}_P^{\,(\mathrm{NEW})}\left(\vec{r}_P^{\,(\mathrm{OLD})}\right) \tag{4.42}$$

where $\vec{r}_P^{\,(\mathrm{NEW})}$ and $\vec{r}_P^{\,(\mathrm{NEW})}$ denote two vector expressions of the same position P in the old and the new coordinate systems, respectively. As is shown here, we label the superscirpts to specify the coordinate system adopted while the subscripts to identify the point.

Now, we expand this relation around the coordinate origin of the old coordinate system as

$$\vec{r}_P^{\,(\mathrm{NEW})} = \vec{r}_O^{\,(\mathrm{NEW})} + \left(\partial \vec{r}_P^{\,(\mathrm{NEW})} \vec{r}_P^{\,(\mathrm{OLD})}\right)_{\vec{r}_P^{\,(\mathrm{OLD})}=0} \vec{r}_P^{\,(\mathrm{OLD})}$$
$$+\frac{1}{2}\left(\frac{\partial^2 \vec{r}_P^{\,(\mathrm{NEW})}}{\partial \vec{r}_P^{\,(\mathrm{OLD})} \partial \vec{r}_P^{\,(\mathrm{OLD})}}\right)_{\vec{r}_P^{\,(\mathrm{OLD})}=0} \vec{r}_P^{\,(\mathrm{OLD})} \vec{r}_P^{\,(\mathrm{OLD})} + \cdots \tag{4.43}$$

where

$$\vec{r}_O^{(\mathrm{NEW})} = \vec{r}_P^{(\mathrm{NEW})}(0) \tag{4.44}$$

In most cases, we deal with linear transformations only. Namely, we will ignore the second and higher order terms in the above expansion. Also, we assume that the orthogonality of basis vectors, which we call the coordinate triad, is reserved through the coordinate transformations. Further, we assume that the norm of difference of any pair of position vectors, which is no other than the relative distance between them, is unchanged by the coordinate transformations. Then, the matrix

$$\mathcal{R} \equiv \left(\partial \vec{r}_P^{(\mathrm{NEW})} \vec{r}_P^{(\mathrm{OLD})}\right)_{\vec{r}_P^{(\mathrm{OLD})}=0} \tag{4.45}$$

is orthonormal, and therefore, is a rotation matrix. In other words, the above coordinate transformation is rewritten as

$$\vec{r}_P^{(\mathrm{NEW})} = \vec{r}_O^{(\mathrm{NEW})} + \mathcal{R}\vec{r}_P^{(\mathrm{OLD})} \tag{4.46}$$

Remark that this is also rewritten as

$$\vec{r}_P^{(\mathrm{NEW})} = \mathcal{R}\left(\vec{r}_P^{(\mathrm{OLD})} - \vec{r}_O^{(\mathrm{OLD})}\right) \tag{4.47}$$

where

$$\vec{r}_O^{(\mathrm{NEW})} = -\mathcal{R}\vec{r}_O^{(\mathrm{OLD})} \tag{4.48}$$

Namely, the coordinate transformation is expressed as the combination of two parts; the shift of coordinate origin, and the rotation of the orientation of the coordinate triad, i.e. the orthonormal basis vectors. Thus, in naming the reference frame, we attach two adjectives specifying the coordinate origin and the orientation such as

$$\vec{r}_{\mathrm{Jupiter}}^{(\mathrm{SSB-EQ})}$$

which means the position vector of the Jupiter in the solar-system barycentric (SSB-) equatorial (EQ) coordinate system.

4.7.2 Coordinate Triad

To simplify the situation, let us ignore the shift of coordinate origin for the moment. In that case, only the reference frame, i.e. the orientation of the coordinate system, is to be discussed.

Mathematically speaking, the reference frame is defined as a set of coordinate triad, i.e. the trio of unit vectors which are orthogonal to each other;

$$\mathcal{E} = (\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3) \tag{4.49}$$

where

$$|\vec{e}_j| = 1, \qquad \vec{e}_i \cdot \vec{e}_j = \delta_{ij} \tag{4.50}$$

The relation between two coordinate triads, $\mathcal{E}$ and $\mathcal{F}$, is given by a matrix;

$$\mathcal{E} = \mathcal{R}\mathcal{F} \tag{4.51}$$

where

$$\mathcal{F} = \left(\vec{f}_1 \ \vec{f}_2 \ \vec{f}_3 \right). \tag{4.52}$$

Consider the expressions of a certain vector $\vec{r}$ in these two coordinate triads;

$$\begin{aligned} \vec{r} &= r_1^{(E)} \vec{e}_1 + r_2^{(E)} \vec{e}_2 + r_3^{(E)} \vec{e}_3 \\ &= r_1^{(F)} \vec{f}_1 + r_2^{(F)} \vec{f}_2 + r_3^{(F)} \vec{f}_3 \end{aligned} \tag{4.53}$$

Thus thee components of the rotation matrix $\mathcal{R}$ is expressed as a dyadic of basis vectors as

$$R_{ij} = \vec{e}_i \cdot \vec{f}_j \tag{4.54}$$

4.7.3 Hierarchy of Reference Frames

There are a variety of reference frames. However, we see a tree structure among them. At the very bottom of it, the reference frame covering the whole universe, which we denote *Celestial Reference Frame*. Unfortunately, we have very limited knowledge on the three dimensional structure of local cluster of galaxies, local cluster of cluster of galaxies, and so on. Next placed is the galactic reference frame. Then comes the solar system barycentric reference system, which is the basic reference frame on which most astrometric observations are expressed finally. Another important reference frame is the pair of geocentric and terrestrial reference frames.

Chapter 5

SIGNAL PROPAGATION

5.1 Propagation of Electromagnetic Wave

So far, the signal dealt in astrometry has been mainly the electromagnetic wave only. Since its wavelength is much shorter than the length scale considered[1] , we adopt the approximation of geometric optics. In other words, we introduce the particle approximation and regard the electromagnetic wave as photon.

5.1.1 One-Way Propagation of Light

As the simplest case, let us consider the one-way propagation of light.

In the absence of general relativistic effects, we can assume that a photon suffers no gravitational effects[2] . Thus its motion is linear in vacuum[3] ;

$$\vec{R}(t) = \vec{R}_0 + \vec{V}_0(t - t_0), \qquad \vec{V}(t) = \vec{V}_0. \tag{5.1}$$

where $\vec{R}_0$ and $\vec{V}_0$ are the position and velocity of the photon at the initial epoch t_0, respectively. Remark that its speed is large as

$$\left|\vec{V}(t)\right| = \left|\vec{V}_0\right| \sim c \tag{5.2}$$

Now, the usage of Eq.(5.1) seems simple; just to compute $\vec{R}(t)$ and $\vec{V}(t)$ as functions of t when $(t_0, \vec{R}_0, \vec{V}_0)$ is given. However, the things do not go so straightforward.

[1] For example, the typical length scale, 1 AU $\sim 1.5 \times 10^{11}$m is roughly 10^{18} times larger than the wavelength of visual light.

[2] Of course, we can introduce the Newtonian gravitational effects into the motion of photon. However, the effects are quite small and errorneous. Proper treatments of them should be done in the framework of general relativity, therefore beyond the scope of this treatise.

[3] In most cases, we can regard the light path is in vacuum. The exception is dealt in the section of refraction

5.1.2 Equation of Light Time

From the viewpoint of observation, we know only limited information in analysing the signal propagation. Usually, the direct observables in the case of one-way propagation are

- the arrival time, t_1,
- the direction of incoming light referred to the frame where the observer is at rest, $\vec{d}'$, and
- the wavelength of incoming light, λ'.

Also, we usually know the motions of the source, from which the photon emitted, and the observer, at which the photon arrived, as functions of time. However, we do not know the time of emission t_0 nor the initial velocity vector of the photon $\vec{V}_0$. Thus, the expression Eq.(5.1) should be considered as an equation to be solved for[4] .

Assume that a photon left the source, S, at the time t_0, and arrived at the observer, E, at the time t_1. Also assume that we know the motion of the source, $\vec{r}_S(t)$, and of the observer, $\vec{r}_E(t)$, in advance. As for the reference frame, we adopt the inertial one comoving with the solar system barycenter. Then, Eq.(5.1) becomes

$$\vec{V}_0(t_1 - t_0) = \vec{R}\,(t_1) - \vec{R}_0 = \vec{r}_E\,(t_1) - \vec{r}_S\,(t_0) \tag{5.3}$$

Unfortunately, the initial velocity of the photon, $\vec{V}_0$, is unknown. If the source is rest in the inertial reference frame adopted, the magnitude of the initial velocity of the photon emitted from the source is equal to the speed of light in vacuum as

$$\left|\vec{V}_0\right| = c \tag{5.4}$$

However, the source is generally moving in the reference frame adopted. Thus, we should modify this expression by taking into account[5] the velocity component of the source toward

[4] In other words, the propagation of light is not solved as the initial value problem but as the boundary value problem, or more precisely speaking, as the final value problem.

[5] As a result, the magnitude of the initial velocity of the photon emitted can be larger than c in this framework. Although this seems inadequate in the relativistic viewpoint, it is valid in the Newtonian sense. In fact, if we do not include this effect, the result becomes incompatible with the Galilei's principle of relativity, which claims that the physical phenomena including this light propagation must be invariant with respect to the choice of the inertial frame of reference. Consider the following three inertial frames; 1) the inertial frame where botht of the observer and the source is moving, 2) the inertial frame which moving relative to the first reference frame with the constant velocity $\vec{v}_S\,(t_0)$, and 3) the frame which moving relative to the first reference frame with the constant velocity $\vec{v}_E\,(t_1)$. Describe the equation of light in these frames, solve it, compute the final light direction to be observed by taking the aberrational effect into account, and evaluate the differences among them. The correct formulation, which is given here, should provide the same light direction vector.

the direction of light emitted as

$$\left|\vec{V}_0\right| = c + \frac{\vec{V}_0 \cdot \vec{v}_0}{\left|\vec{V}_0\right|}. \tag{5.5}$$

where

$$\vec{v}_0 \equiv \vec{v}_S\left(t_0\right) \tag{5.6}$$

is the velocity of the source at the start time. By cancelling the unknown vector $\vec{V}_0$ from this and Eq.(5.3), we obtain an equation connecting two times t_1 and t_0 as

$$\left(c - \frac{\vec{v}_0 \cdot \vec{r}_{01}}{r_{01}}\right)\left(t_1 - t_0\right) = r_{01} \tag{5.7}$$

where

$$\vec{r}_{01} \equiv \vec{r}_S\left(t_0\right) - \vec{r}_E\left(t_1\right), \qquad r_{01} \equiv \left|\vec{r}_{01}\right| \tag{5.8}$$

If we know the end time t_1 only, this is regarded as an equation to be solved with respect to the start time t_0. Or, instead, we may regard it as the equation to obtain the time difference

$$\tau \equiv t_1 - t_0 \tag{5.9}$$

which is named as the light travel time, or the *light time* in short. Namely the equation is rewritten as

$$V(\tau)\tau = r(\tau) \tag{5.10}$$

This is called as the *equation of light time*. Here

$$V(\tau) \equiv c + \vec{v}(\tau) \cdot \vec{d}(\tau), \tag{5.11}$$

denotes the speed of the photon,

$$\vec{d}(\tau) \equiv \frac{\vec{r}(\tau)}{r(\tau)}, \tag{5.12}$$

does the direction of the incoming photon seen from the observer. Also

$$\vec{r}(\tau) \equiv \vec{r}_S\left(t_1 - \tau\right) - \vec{r}_E\left(t_1\right), \tag{5.13}$$

is the difference of the position vectors of the source at the start time and that of the observer at the end time, while

$$r(\tau) = \left|\vec{r}(\tau)\right|, \tag{5.14}$$

denotes its magnitude, and

$$\vec{v}(\tau) \equiv \frac{\mathrm{d}\vec{r}(\tau)}{\mathrm{d}\tau} = -\vec{v}_0 = -\vec{v}_S\,(t_1 - \tau) \tag{5.15}$$

does its time derivative, respectively.

Remark that $\vec{r}_E\,(t_1)$ is a known vector and is constant with respect to the new variable τ. Thus the motion of observer doesn't matter with the equation of light time itself[6] .

5.1.3 Solving Equation of Light Time

The equation of light time

$$f_{LT}(\tau) \equiv V(\tau)\tau - r(\tau) = 0 \tag{5.16}$$

is nonlinear. Generally its solution is not obtained in a closed form. Thus, it is usually solved numerically and iteratively; namely solved by the combination of a starting formula and a correction procedure.

As for the correction procedure, we prefer the Newton method;

$$\tau \to \tau^*(\tau) \equiv \tau - \frac{f_{LT}(\tau)}{f'_{LT}(\tau)} = \frac{r(\tau) - r'(\tau)\tau + V'(\tau)\tau^2}{V(\tau) - r'(\tau) + V'(\tau)\tau} \tag{5.17}$$

Here

$$r'(\tau) \equiv \frac{\mathrm{d}r(\tau)}{\mathrm{d}\tau} = \vec{v}(\tau)\cdot\vec{d}(\tau), \tag{5.18}$$

$$V'(\tau) \equiv \frac{\mathrm{d}V(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 r(\tau)}{\mathrm{d}\tau^2} = \vec{a}(\tau)\cdot\vec{d}(\tau) + \vec{v}(\tau)\cdot\vec{e}(\tau) \tag{5.19}$$

and the derivatives are given[7] as

$$\vec{a}(\tau) \equiv \frac{\mathrm{d}\vec{v}(\tau)}{\mathrm{d}\tau} = -\left(\frac{\mathrm{d}\vec{v}_S}{\mathrm{d}t}\right)(t_1 - \tau)\,, \tag{5.20}$$

and

$$\vec{e}(\tau) \equiv \frac{\mathrm{d}\vec{d}(\tau)}{\mathrm{d}\tau} = \frac{\vec{v}(\tau) - \left\{\vec{d}(\tau)\cdot\vec{v}(\tau)\right\}\vec{d}(\tau)}{r(\tau)} \tag{5.21}$$

[6] The motion of the observer palys the key role in the aberration.

[7] See the Appendix for the differential formulas.

Remark that the second term of $V(\tau)$ cancels with $r'(\tau)$ so that the Newton corrector becomes simpler as

$$\tau^*(\tau) = \frac{r(\tau) - r'(\tau)\tau + V'(\tau)\tau^2}{c + V'(\tau)\tau} \tag{5.22}$$

As for the initial guess, we start from the value $\tau = 0$, which is the solution in the limit of $c \to \infty$.

If the source is within the solar system, then $\vec{v}_S/c$ is small as of the order of 10^{-3} at most[8] . Thus, usually only one correction is enough in that case. Then, an approximate solution is given by applying the Newton correction once as

$$\tau \approx \tau_1 = \frac{r(0)}{c} = \frac{r_{ES}(t_1)}{c} \tag{5.23}$$

where

$$r_{ES}(t_1) = |\vec{r}_E(t_1) - \vec{r}_S(t_1)| \tag{5.24}$$

is the distance bewteen the source and the observer at the end time t_1, which is called the *simultaneous distance* or *geometrical distance*. If we proceed to the next approximation,

$$\tau^*(\tau_1) = \frac{r(\tau_1) - r'(\tau_1)\tau_1 + V'(\tau_1)\tau_1^2}{c + V'(\tau_1)\tau_1} = \frac{r(0) + (r''(0)/2)\tau_1^2 + O(\tau^3)}{c + r''(0)\tau_1 + O(\tau^3)} \tag{5.25}$$

Thus the first approximate solution, τ_1, is correct up to the order of v/c. In computing the next stage of approximation correctly, we should take the general relativistic effects into account. Therefore, we stop here.

5.2 Light Direction

Once the equation of light time is solved, the signal propagation is completely known. Then the other observable, the light direction $\vec{d}$, the direction vector of the incoming light at the observer, is obtained as

$$\vec{d} = \vec{d}(\tau) = \frac{-\vec{V}_0}{\left|\vec{V}_0\right|} = \frac{\vec{r}_{01}}{r_{01}} \tag{5.26}$$

where

$$\vec{r}_{01} = \vec{r}_S(t_0) - \vec{r}_E(t_1), \qquad r_{01} = |\vec{r}_{01}| \tag{5.27}$$

[8] The extreme case would be the Mercury. Its orbital velocity is around 300 km/s, which corresponds to 0.1% of the speed of light in vacuum, c

Remark that the direction vector described here is that viewed in the inertial frame adopted, namely the direction vector seen from another observer comoving with the solar system barycenter. The direction vector seen by the moving observer differs from this $\vec{d}$. The deviation[9] will be discussed in the followings.

5.2.1 Aberration

The velocity of the light is finite. Thus the direction of an incoming photon seen from a moving observer changes a little with respect to the direction seen from another observer being rest. This phenomenon is just the same as the declination of raindrop tracks. The tracks of raindrops[10] on side windows of a moving car show lines declining somewhat from the vertical lines, which should be the tracks on the windows of a car on rest. This difference in light direction is called as the *aberration*[11] .

Assume that the observer moves at the velocity $\vec{v}_E(t)$ in the inertial reference frame adopted. Consider changing the reference frame to an inertial frame which contacted[12] with the observer at the end time, t_1. In other words, we transfer to a new inertial frame which moves relative to the old one with the constant velocity

$$\vec{v}_1 \equiv \vec{v}_E\left(t_1\right). \tag{5.28}$$

Then, the light direction in the new reference frame is obtained by changing the velocity of the photon from that referred to the old inertial reference frame, $\vec{V}_0$, to that referred to the new inertial reference frame, $\vec{V}_0 - \vec{v}_1$. Therefore, the light direction in the old reference frame, $\vec{d}$, is transformed to that in the new reference frame, $\vec{d'}$, as

$$\vec{d'} = \frac{-\left(\vec{V}_0 - \vec{v}_1\right)}{\left|\vec{V}_0 - \vec{v}_1\right|} = \frac{\vec{d} + \vec{v}_1/V_0}{\left|\vec{d} + \vec{v}_1/V_0\right|} \approx \vec{d} + \frac{1}{c}\left[\vec{v}_1 - \left(\vec{d}\cdot\vec{v}_1\right)\vec{d}\right] \tag{5.29}$$

where the second and higher order effects are ignored.

In the case when the observer is rest on the Earth, the magnitude of $\vec{v}_1/c$ is $10^{-4} \approx 20''$. Thus the second order effect is of the order of mas[13] . If we write the angle between the light direction $\vec{d}$ and the moving direction $\vec{v}_1$ as θ, namely

$$\vec{d}\cdot\vec{v}_1 = v_1 \cos\theta, \qquad v_1 = \left|\vec{v}_1\right| \tag{5.30}$$

[9] Named as the *abberation*.
[10] When no wind blows.
[11] The aberration was discovered by Bradley.
[12] It seems a good choice to adopt the reference frame which is moving with the observer. However, the motion of observer is generally an accelerated one. Thus, the reference fram moving with the observer is not inertial. Therefore, we cannot adopt it as the new reference frame.
[13] If the second order effects are no more negligiblly small, one should use the full formula of general relativity. It is beyond the scope of this treatise. Thus we stop here.

then the above vector expression of the aberration is rewritten as

$$\theta' \approx \theta - \frac{v_1}{c} \sin\theta < \theta \tag{5.31}$$

Namely the angle between the light direction and the moving direction seems narrower if seen from a moving observer[14] .

Now, in order to see the actual effects of aberration, we will consider the approximations of two special but practically important cases in the followings.

5.2.2 Annual Aberration

Assume that the observer is at the geocenter. Simplify the orbital motion of the Earth as a constant-speed circular motion on the ecliptic.

Let us compute the effect of aberration of a star; namely the deviation of direction vector between that seen from the moving observer and that seen from the Sun[15] . In this case, it is better to take the ecliptic coordinate system. Then the direction vectors of the star with and without the aberrational effects are written respectively as

$$\vec{d'} = \begin{pmatrix} \cos\beta' \cos\lambda' \\ \cos\beta' \sin\lambda' \\ \sin\beta' \end{pmatrix}, \qquad \vec{d} = \begin{pmatrix} \cos\beta \cos\lambda \\ \cos\beta \sin\lambda \\ \sin\beta \end{pmatrix} \tag{5.32}$$

and the veclocity of the observer around the Sun is expressed as

$$\vec{v}_1 = V\vec{e}_1 = V \begin{pmatrix} -\sin L \\ \cos L \\ 0 \end{pmatrix} \tag{5.33}$$

where V is the orbital velocity of the Earth around the Sun[16] , $\vec{e}_1$ is its direction vector, and L is the mean longitude of the Earth, which is simply a linear function fo time in this approximation.

Thus, the deviation of direction vector is approximately written as

$$\Delta\vec{d'} \equiv \vec{d'} - \vec{d} \approx \frac{1}{c} \left[\vec{v}_1 - \left(\vec{d} \cdot \vec{v}_1 \right) \vec{d} \right], \tag{5.34}$$

This is rewritten in the component-wise form as

$$\Delta\beta' \begin{pmatrix} -\sin\beta\cos\lambda \\ -\sin\beta\sin\lambda \\ \cos\beta \end{pmatrix} + \Delta\lambda' \begin{pmatrix} -\cos\beta\sin\lambda \\ \cos\beta\cos\lambda \\ 0 \end{pmatrix}$$

[14] This is the well-known *starbow* effect, which appears frequently in an SF movie scene where a starship moves very quickly by a warp or other SciFi-like technique.

[15] Rigorously speaking, the reference direction should be that seen from the barycenter of the solar system again. However, we ignore the mass of the Earth and other planets for simplicity.

[16] We assume V is constant here. Its numerical value is roughly 30 km/s.

$$\approx \kappa \begin{pmatrix} -\sin L - D\cos\beta\cos\lambda \\ \cos L - D\cos\beta\sin\lambda \\ -D\sin\beta \end{pmatrix} \tag{5.35}$$

where

$$\kappa \equiv \frac{V}{c} \tag{5.36}$$

and

$$\Delta\beta' = \beta' - \beta, \qquad \Delta\lambda' = \lambda' - \lambda, \qquad D \equiv \vec{d}\cdot\vec{e}_1 = -\cos\beta\sin(L-\lambda) \tag{5.37}$$

The quantity κ has been called as the *aberration constant*, since it can be thought as a constant within this approximation. The magnitude of κ is roughly

$$\kappa = \frac{V}{c} \sim \frac{3\times 10^4 \mathrm{m/s}}{3\times 10^8 \mathrm{m/s}} \sim 10^{-4} \sim 20'' \tag{5.38}$$

By solving the above approximate equation with respect to $\Delta\beta'$ and $\Delta\lambda'$, we obtain an approximate formula of aberrational variations of ecliptic coordinates of the star due to the orbital motion of the Earth as

$$\Delta\beta' \approx \kappa\sin\beta\sin(L-\lambda), \tag{5.39}$$

$$(\cos\beta)\,\Delta\lambda' \approx -\kappa\cos(L-\lambda) \tag{5.40}$$

By elliminating the factor $L-\lambda$ which is a linear function of time, we obtain an approximate relation connecting the aberrations in ecliptic longitude and in ecliptic latitude as

$$\left(\frac{\xi}{\kappa}\right)^2 + \left(\frac{\eta}{\kappa\sin\beta}\right)^2 = 1 \tag{5.41}$$

where

$$\xi \equiv (\cos\beta)\,\Delta\lambda, \qquad \eta \equiv \Delta\beta \tag{5.42}$$

are the coordinates on the plane tangential to the unit sphere at (λ, β). This is the equation of an ellipse on the tangential coordinate (ξ, η). The ellipse is called as the *aberrational ellipse*. The semi-major axis of the ellipse is the annual aberration constant, κ, itself while the axis ratio is the same as the sine of ecliptic latitude $\sin\beta$. In other words, the aberrational ellipse becomes a circle for the stars in the directions of the ecliptic poles, and becomes a straight line when the star is on the ecliptic. Since the period of the time variation of this effect is one year, the aberration discussed here is called as the *annual aberration*. Remark that the phase of annual aberration is around 90° offset from that of the annual parallax, which will be discussed later.

5.2.3 Diurnal Aberration

Another aberrational effect appears when the observer is rest on the Earth.

Assume that the observer is on the surface of the Earth. Ignore the height of the observer from the reference spheroid, the eccentricity of the Earth spheroid, and the fluctuation of the spin axis of the Earth. Then the motion of the observer is approximated as a constant rotation around the pole axis.

Consider to compute the effect of aberration due to the difference of the observer's velocity vector referred to the geocenter. In this case, it is better to take the equatorial coordinate system. The direction vectors of an celetial object with and without the aberrational effects are written respectively as

$$\vec{d'} = \begin{pmatrix} \cos\delta' \cos\alpha' \\ \cos\delta' \sin\alpha' \\ \sin\delta' \end{pmatrix}, \qquad \vec{d} = \begin{pmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{pmatrix} \tag{5.43}$$

and the velocity of the observer around the geocenter is expressed as

$$\vec{v}_1 = V' \cos\phi \vec{e}_1 = V' \cos\phi \begin{pmatrix} -\sin\Theta \\ \cos\Theta \\ 0 \end{pmatrix} \tag{5.44}$$

where ϕ is the latitude of the observer, Θ is the local sidereal time at the longitude of the observer, and V' is the equatorial velocity of the Earth, which is expressed as

$$V' = \omega R \tag{5.45}$$

by using the angular velocity of the Earth rotation, ω. Remark that Θ is a linear function fo time in this approximation.

Then the deviation of direction vector is approximately written as

$$\Delta\vec{d'} \equiv \vec{d'} - \vec{d} \approx \frac{1}{c}\left[\vec{v}_1 - \left(\vec{d} \cdot \vec{v}_1\right)\vec{d}\right] \tag{5.46}$$

This is rewritten in the component-wise form as

$$\begin{aligned} &\Delta\delta' \begin{pmatrix} -\sin\delta\cos\alpha \\ -\sin\delta\sin\alpha \\ \cos\delta \end{pmatrix} + \Delta\alpha' \begin{pmatrix} -\cos\delta\sin\alpha \\ \cos\delta\cos\alpha \\ 0 \end{pmatrix} \\ &\qquad \approx \kappa' \cos\phi \begin{pmatrix} -\sin\Theta - D'\cos\delta\cos\alpha \\ \cos\Theta - D'\cos\delta\sin\alpha \\ -D'\sin\delta \end{pmatrix} \end{aligned} \tag{5.47}$$

where

$$\Delta\delta' = \delta' - \delta, \qquad \Delta\alpha' = \alpha' - \alpha, \tag{5.48}$$

and

$$D' \equiv \vec{d} \cdot \vec{e}_1 = -\cos\delta \sin(\Theta - \alpha) \tag{5.49}$$

The quantity

$$\kappa' = \frac{V'}{c} = \frac{\omega R}{c} \tag{5.50}$$

is called as the *diurnal aberration constant*. Remark that the magnitude of diurnal aberration is proportional to $\cos\phi$. This is the latitude effect in the diurnal aberration.

By solving the above approximate equation with respect to $\Delta\delta$ and $\Delta\alpha$, we obtain an approximate formula of aberrational variations of equatorial coordinates of the star due to the Earth rotation as

$$\Delta\delta = \kappa' \cos\phi \sin\delta \sin(\Theta - \alpha), \tag{5.51}$$

$$\cos\delta\Delta\alpha = \kappa' \cos\phi \cos(\Theta - \alpha) \tag{5.52}$$

By elliminating the factor $\Theta - \alpha$, which is a linear function of time, we obtain a relation connecting the aberrations in right ascension and in declination as

$$\left(\frac{\xi'}{\kappa''}\right)^2 + \left(\frac{\eta'}{\kappa'' \sin\delta}\right)^2 = 1 \tag{5.53}$$

where

$$\xi' \equiv \cos\delta\Delta\alpha, \qquad \eta' \equiv \Delta\delta \tag{5.54}$$

are the coordinates on the plane tangential to the unit sphere at (δ, α) and

$$\kappa'' \equiv \kappa' \cos\phi, \tag{5.55}$$

This is the equation of another aberrational ellipse. Since the period of the time variation of this effect is one day, the aberration discussed here is called as the *diurnal aberration*. The treatment given here is of approximate nature. Therefore, one may use the rigorous formulation, which begins with the solution of equation of light time.

5.3 Parallax

When the distance to the source is large such as the case of stars, the equation of light time becomes meaningless. We should start from the expression of the light direction.

$$\vec{d} = \frac{\vec{r}_{01}}{r_{01}} \tag{5.56}$$

Here

$$\vec{r}_{01} = \vec{r}_0 - \vec{r}_1, \qquad r_{01} = |\vec{r}_{01}| \tag{5.57}$$

while $\vec{r}_0$ is the position vector of the source at the emission time and $\vec{r}_1$ is the position vector of the observer at the receiving time. As far as the direction vector is dealt, not $\vec{r}_0$ itself but the pair $(\vec{d}, 1/r_0)$ is easier to handle. Here

$$\vec{d}_0 = \frac{\vec{r}_0}{r_0}, \qquad r_0 = |\vec{r}_0| \tag{5.58}$$

is the unit vector and $1/r_0$ is the inverse distance.

Now the light direction $\vec{d}$ is expanded with respect to $1/r_0$ as

$$\vec{d} = \frac{\vec{r}_0 - \vec{r}_1}{|\vec{r}_0 - \vec{r}_1|} = \frac{\vec{d}_0 - \vec{r}_1/r_0}{\left|\vec{d}_0 - \vec{r}_1/r_0\right|} \approx \vec{d}_0 - \frac{1}{r_0}\left[\vec{r}_1 - \left(\vec{d}_0 \cdot \vec{r}_1\right)\vec{d}_0\right], \tag{5.59}$$

where the second and higher order terms are ignored[17] . In the case of stars, the magnitude of $\vec{r}_1/r_0$ is of the order of arcsecond at most. Thus the second order effects are negligiblly small as of the order of microarcsecond or less.

In expressing the inverse distance, however, we usually nondimensionize it as

$$p \equiv \sin^{-1}\frac{A}{r_0} \approx \frac{A}{r_0} \tag{5.60}$$

where A is AU. We denote p as the *parallax*[18] of the source[19] .

Now, in order to see the actual effects of parallax, we will consider the approximations of two special but practically important cases in the followings.

[17] Rather, if the inverse distance is not sufficiently small, that means the distance dealt should be regard as finite. In that case, one must deal the position vector of the source in a three dimensional form, and therefore, should start from the equation of light time.

[18] In astronomy, the word *parallax* has been used loosely. It is frequently used when the word *distance* should be. Sometimes an adjective explaining the computing method is attached to the word *parallax*. The typical usage is the statistical parallax, which means, rigorously the (inverse) distance determined from statistical methods. In this context, the parallax here, which is geometrically defined, is frequently called as *trigonometric parallax* simply because the trigonometry is used (See the appearance of $\sin^{-1}$). Sometimes the trigonometric parallax is called as the annual parallax, since the period of its main time variation is one solar year.

[19] Originally the symbol π has been used. We prefer p since π is confusing with $3.141592\cdots$

5.3.1 Annual Parallax

Assume that the observer is at the geocenter. Simplify the orbital motion of the Earth as a constant-speed circular motion on the ecliptic.

Let us compute the effect of parallax of a star; namely the deviation of direction vector between that seen from the observer and that seen from the Sun[20] . In this case, it is better to take the ecliptic coordinate system. Then the direction vectors of the star with and without the parallactic effects are written respectively as

$$\vec{d} = \begin{pmatrix} \cos\beta\cos\lambda \\ \cos\beta\sin\lambda \\ \sin\beta \end{pmatrix}, \qquad \vec{d_0} = \begin{pmatrix} \cos\beta_0\cos\lambda_0 \\ \cos\beta_0\sin\lambda_0 \\ \sin\beta_0 \end{pmatrix} \tag{5.61}$$

and the position of the observer with respect to the Sun is expressed as

$$\vec{r_1} = A\vec{d_1} = A\begin{pmatrix} \cos L \\ \sin L \\ 0 \end{pmatrix}, \tag{5.62}$$

and $\vec{d_1}$ is its direction vector, and L is the mean longitude of the Earth, which is simply a linear function fo time in this approximation.

Thus, the deviation of direction vector is approximately written as

$$\Delta\vec{d} \equiv \vec{d} - \vec{d_0} \approx \frac{-1}{r_0}\left[\vec{r_1} - \left(\vec{d_0}\cdot\vec{r_1}\right)\vec{d_0}\right] \tag{5.63}$$

This is rewritten in the component-wise form as

$$\begin{aligned} \Delta\beta\begin{pmatrix} -\sin\beta_0\cos\lambda_0 \\ -\sin\beta_0\sin\lambda_0 \\ \cos\beta_0 \end{pmatrix} + \Delta\lambda\begin{pmatrix} -\cos\beta_0\sin\lambda_0 \\ \cos\beta_0\cos\lambda_0 \\ 0 \end{pmatrix} \\ \approx p\begin{pmatrix} C\cos\beta_0\cos\lambda_0 - \cos L \\ C\cos\beta_0\sin\lambda_0 - \sin L \\ C\sin\beta_0 \end{pmatrix} \end{aligned} \tag{5.64}$$

where

$$\Delta\beta = \beta - \beta_0, \qquad \Delta\lambda = \lambda - \lambda_0, \qquad C \equiv \vec{d_0}\cdot\vec{d_1} = \cos\beta_0\cos\left(L - \lambda_0\right) \tag{5.65}$$

[20] Rigorously speaking, the reference direction should be that seen from the barycenter of the solar system However, we ignore the mass of the Earth and other planets for simplicity.

By solving the above approximate equation with respect to $\Delta\beta$ and $\Delta\lambda$, we obtain the approximate formula of parallactic variations of ecliptic coordinates of the star due to the orbital motion of the Earth as

$$\Delta\beta \approx p \sin\beta_0 \cos(L - \lambda_0), \tag{5.66}$$

$$(\cos\beta_0)\, \Delta\lambda \approx -p \sin(L - \lambda_0) \tag{5.67}$$

By elliminating the factor $L - \lambda_0$ which is a linear function of time, we obtain an approximate relation connecting the parallaxes in ecliptic longitude and in ecliptic latitude as

$$\left(\frac{\xi}{p}\right)^2 + \left(\frac{\eta}{p \sin\beta_0}\right)^2 = 1 \tag{5.68}$$

where

$$\xi \equiv (\cos\beta_0)\, \Delta\lambda, \qquad \eta \equiv \Delta\beta \tag{5.69}$$

are the coordinates on the plane tangential to the unit sphere at (β_0, λ_0). This is the equation of an ellipse on the tangential coordinate (ξ, η). The ellipse is called as the *parallactic ellipse*. The semi-major axis of the ellipse is the parallax, p, itself while the axis ratio is the same as the sine of ecliptic latitude $\sin\beta_0$. In other words, the parallactic ellipse becomes a circle for the stars in the directions of the ecliptic poles, and becomes a straight line when the star is on the ecliptic plane. Since the period of the time variation of this effect is one year, the parallax discussed here is called also as the *annual parallax* Remark that the phase of annual parallax is around 90° offset from that of the annual aberration. Also note that, even in the same direction, the amplitude of the parallax does differ star by star while that of the aberration is the same.

5.3.2 Diurnal Parallax

Another parallactic effect appears when the observer is rest on the Earth.

Assume that the observer is on the surface of the Earth. Ignore the height of the observer from the reference spheroid, the eccentricity of the Earth spheroid, and the fluctuation of the spin axis of the Earth. Then the position of the observer is always offset from the geocenter by the amount of the radius of the Earth, although its vector components approximately vary with time by a constant rotation around the pole axis.

Consider to compute the effect of parallax due to the difference of the observer's position between that on the Earth and that at the geocenter. In this case, it is better to take the equatorial coordinate system. The direction vectors of an celetial object with and without the parallactic effects are written respectively as

$$\vec{d} = \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix}, \qquad \vec{d_0} = \begin{pmatrix} \cos\delta_0\cos\alpha_0 \\ \cos\delta_0\sin\alpha_0 \\ \sin\delta_0 \end{pmatrix} \tag{5.70}$$

and the motion of the observer around the geocenter is expressed as

$$\vec{r}_1 = a\vec{d_1} = a \begin{pmatrix} \cos\phi\cos\Theta \\ \cos\phi\sin\Theta \\ \sin\phi \end{pmatrix} \tag{5.71}$$

where a is the radius[21] of the Earth, ϕ is the latitude[22] of the observer, Θ is the local sidereal time[23] at the longitude of the observer. Remark that Θ denote actually the angle of the Earth rotation, which is simply a linear function fo time in this approximation.

Then the deviation of direction vector is approximately written as

$$\Delta\vec{d} \equiv \vec{d} - \vec{d_0} \approx -\frac{1}{r_0}\left[\vec{r}_1 - \left(\vec{d_0}\cdot\vec{r}_1\right)\vec{d_0}\right] \tag{5.72}$$

This is rewritten in the component-wise form as

$$\Delta\delta \begin{pmatrix} -\sin\delta_0\cos\alpha_0 \\ -\sin\delta_0\sin\alpha_0 \\ \cos\delta_0 \end{pmatrix} + \Delta\alpha \begin{pmatrix} -\cos\delta_0\sin\alpha_0 \\ \cos\delta_0\cos\alpha_0 \\ 0 \end{pmatrix}$$

$$\approx p' \begin{pmatrix} C'\cos\delta_0\cos\alpha_0 - \cos\phi\cos\Theta \\ C'\cos\delta_0\sin\alpha_0 - \cos\phi\sin\Theta \\ C'\sin\delta_0 - \sin\phi \end{pmatrix} \tag{5.73}$$

where

$$\Delta\delta = \delta - \delta_0, \qquad \Delta\alpha = \alpha - \alpha_0, \tag{5.74}$$

and

$$C' \equiv \vec{d_0}\cdot\vec{d_1} = \cos\delta_0\cos\phi\cos\left(\Theta - \alpha_0\right) + \sin\delta_0\sin\phi \tag{5.75}$$

[21] We ignored the difference between the equatorial radius, a_e, and the polar radius, a_p, here.
[22] Also we ignored the difference between the geographic latitude, φ, and the geocentric latitude, ϕ, here.
[23] Further we ignored the difference between the mean sidereal time, MST, and the apparent sidereal time, AST, here.

The quantity

$$p' = \sin^{-1} \frac{R}{r_0} \approx \frac{R}{r_0} \tag{5.76}$$

is called as the *horizontal parallax* of the object. The difference between the appearance of this and the annual parallaxes are the existence of the z-term, $\sin\phi$, which is constant with respect to time. The effect caused by this z-component is called as the *latitude effect.*

By solving the above approximate equation with respect to $\Delta\delta$ and $\Delta\alpha$, we obtain the approximate formula of parallactic variations of equatorial coordinates of the star due to the Earth rotation as

$$\Delta\delta = p' \left[\cos\phi \sin\delta_0 \cos(\Theta - \alpha_0) - \sin\phi \cos\delta_0\right], \tag{5.77}$$

$$(\cos\delta_0)\,\Delta\alpha = -p' \cos\phi \sin(\Theta - \alpha_0) \tag{5.78}$$

By cancelling the factor $\Theta - \alpha_0$, which is a linear function of time, we obtain a relation connecting the parallaxes in right ascension and in declination as

$$\left(\frac{\xi'}{p''}\right)^2 + \left(\frac{\eta' - \eta'_0}{p'' \sin\delta_0}\right)^2 = 1 \tag{5.79}$$

where

$$\xi' \equiv (\cos\delta_0)\,\Delta\alpha, \qquad \eta' \equiv \Delta\delta \tag{5.80}$$

are the coordinates on the plane tangential to the unit sphere at (δ_0, α_0) and

$$p'' \equiv p' \cos\phi, \qquad \eta'_0 \equiv -p' \sin\phi \cos\delta_0, \tag{5.81}$$

This is the equation of another parallactic ellipse. Remark that the center of the ellipse is offset by the latitude effect. Since the period of the time variation of this effect is one day, the parallax discussed here is called as the *diurnal parallax.* The diurnal parallax is tiny for the stars. It appears for the solar system objects such as the Moon. However, the treatment given here is of approximate nature. Therefore, one may use the rigorous formulation, which begins with the solution of equation of light time.

5.4 Application of Equation of Light Time

5.4.1 Round Trip Propagation

As an application of the equation of light time, consider the two-way propagation. Namely assume that a photon left a point at the time t_0, arrived at another point at the time t_1, then reached at yet another point at the time t_2. We call the first the start point, S, the second the middle point, M, and the last the end point, E. Also assume that we know the motion of all three points; $\vec{r}_S(t)$, $\vec{r}_M(t)$, and $\vec{r}_E(t)$, in advance. Then, Eq.(5.7) is written for each path as

$$c\left(t_1 - t_0\right) = \left|\vec{r}_M\left(t_1\right) - \vec{r}_S\left(t_0\right)\right|, \qquad c\left(t_2 - t_1\right) = \left|\vec{r}_E\left(t_2\right) - \vec{r}_M\left(t_1\right)\right| \tag{5.82}$$

In general, these two are solved separately with respect to the light times $\sigma_{10} \equiv t_1 - t_0$ and $\sigma_{21} \equiv t_2 - t_1$. More precisely speaking, one must 1) begin with the end time t_2, 2) solve the latter equation of light time to obtain the middle time t_1, and 3) by using thus obtained t_1, solve the former equation of light time to obtain the start time t_0.

However, there is a special case which is interesting and appears frequently in practical observations; a round-trip propagation of light. The typical case is the laser ranging to artificial Earth satellites or to the Moon. The round-trip propagation is defined as the two-way propagation where the start and the end points are the same; S = E. Let us rename the middle point, M, as the target, T, in this case. The basic equations of light time are

$$c\left(t_1 - t_0\right) = \left|\vec{r}_T\left(t_1\right) - \vec{r}_E\left(t_0\right)\right|, \qquad c\left(t_2 - t_1\right) = \left|\vec{r}_E\left(t_2\right) - \vec{r}_T\left(t_1\right)\right| \tag{5.83}$$

Usually, both the start time t_0 and the end time t_2 are observables in this case. Thus, we should regard these as the equation for the middle time t_1.

Remark that the number of equations are two while the number of unknown is one. Namely, these equations are superfluous and the unknown t_1 will be overdetermined. However, this situation is avoided as follows. In practice, we introduce a new variable σ defined as

$$\sigma = t_1 - \frac{t_2 + t_0}{2} \tag{5.84}$$

It is easy to show that the sum of the above two equations is rather insensitive with respect to the new variable σ. Thus, we discard that summation information. Instead, taking the difference of the above two equations, we obtain the equation to be solved, which we call the equation of round-trip light time;

$$f_{RT}(\sigma) \equiv c\sigma - \frac{\left|\vec{r}_T\left(t_1' + \sigma\right) - \vec{r}_E\left(t_0\right)\right| - \left|\vec{r}_E\left(t_2\right) - \vec{r}_T\left(t_1' + \sigma\right)\right|}{2} = 0 \tag{5.85}$$

where

$$t_1' \equiv \frac{t_2 + t_0}{2} \tag{5.86}$$

is the zeroth approximation of the middle time.

Usually, the equation of round-trip light time is solved by the Newton method. The Newton corrector becomes

$$\sigma^{*}(\sigma) \equiv \sigma - \frac{f_{RT}(\sigma)}{f'_{RT}(\sigma)} = \frac{(r_{01} - r_{21})\, r_{01} r_{21} + \sigma \left(r_{01} \vec{r}_{21} + r_{21} \vec{r}_{01}\right) \cdot \vec{v}_1}{2 c r_{01} r_{21} + \left(r_{01} \vec{r}_{21} + r_{21} \vec{r}_{01}\right) \cdot \vec{v}_1} \tag{5.87}$$

where

$$\vec{r}_{01} \equiv \vec{r}_E\left(t_0\right) - \vec{r}_T\left(t'_1 + \sigma\right), \qquad r_{01} \equiv \left|\vec{r}_{01}\right|, \tag{5.88}$$

$$\vec{r}_{21} \equiv \vec{r}_E\left(t_2\right) - \vec{r}_T\left(t'_1 + \sigma\right), \qquad r_{21} \equiv \left|\vec{r}_{21}\right|, \tag{5.89}$$

and

$$\vec{v}_1 \equiv \vec{v}_T\left(t'_1 + \sigma\right). \tag{5.90}$$

As for the initial guess, we start from the value $\sigma = 0$. Then, an approximate solution is given by applying the Newton correction once as

$$\sigma \approx \frac{-\left(r_2 - r_0\right) r_0 r_2}{2 c r_0 r_2 + \left(r_0 \vec{r}_2 + r_2 \vec{r}_0\right) \cdot \vec{v}'_T} \tag{5.91}$$

where

$$\vec{r}_0 \equiv \vec{r}_E\left(t_0\right) - \vec{r}_T\left(t'_1\right), \qquad r_0 \equiv \left|\vec{r}_0\right|, \qquad \vec{r}_2 \equiv \vec{r}_E\left(t_2\right) - \vec{r}_T\left(t'_1\right), \qquad r_2 \equiv \left|\vec{r}_2\right|, \tag{5.92}$$

and

$$\vec{v}'_T \equiv \vec{v}_T\left(t'_1\right). \tag{5.93}$$

This is further approximated as

$$\sigma \approx \frac{\left(t_2 - t_0\right) \vec{d}' \cdot \vec{v}'_E}{2\left(c - \vec{d}' \cdot \vec{v}'_T\right)} \tag{5.94}$$

where

$$\vec{d}' \equiv \frac{-\vec{r}'}{r'}, \qquad \vec{r}' \equiv \vec{r}_E\left(t'_1\right) - \vec{r}_T\left(t'_1\right), \qquad r' \equiv \left|\vec{r}'\right|, \qquad \vec{v}'_E \equiv \vec{v}_E\left(t'_1\right). \tag{5.95}$$

Here we used the expansions

$$r_0 \equiv \vec{r}_1 c\left(t_1 - t_0\right) = r\left(t_1\right) - \vec{d}_1 \cdot \vec{r}\left(t_1\right) + \cdots \tag{5.96}$$

$$c\left(t_2 - t_1\right) = r\left(t_1\right) + \vec{d}_1 \cdot \vec{r}\left(t_1\right) + \cdots \tag{5.97}$$

5.4.2 Equation of Pulse Time Arrival

When the distance to the source is large such as the case of stars, the equation of light time becomes meaningless unless we receive a train of light pulse constantly.

Consider that we are observing such a pulse train

$$t_1^{(0)}, t_1^{(1)}, \cdots, t_1^{(n)}, \cdots,$$

which are the arrival times of the light pulses emitted

$$t_0^{(0)}, t_0^{(1)}, \cdots, t_0^{(n)}, \cdots,$$

For simplicity, we assume that the emission of light pulses are constant;

$$t_0^{(n)} = t_0^{(0)} + nP \tag{5.98}$$

where P is the period of pulse, and n is the integer denoting the order of the pulse in the pulse train. The equation of light time is written for the zeroth and n-th pulses respectively as

$$c\left(t_1^{(0)} - t_0^{(0)}\right) = \left|\vec{r}_E\left(t_1^{(0)}\right) - \vec{r}_S\left(t_0^{(0)}\right)\right|, \tag{5.99}$$

$$c\left(t_1^{(n)} - t_0^{(n)}\right) = \left|\vec{r}_E\left(t_1^{(n)}\right) - \vec{r}_S\left(t_0^{(n)}\right)\right|. \tag{5.100}$$

Taking the difference between the above two equations and by substituting Eq.(5.98) into it, we obtain

$$c\left(t_1^{(n)} - t_1^{(0)} - nP\right) = \left|\vec{r}_E\left(t_1^{(n)}\right) - \vec{r}_S\left(t_0^{(0)} + nP\right)\right| - \left|\vec{r}_E\left(t_1^{(0)}\right) - \vec{r}_S\left(t_0^{(0)}\right)\right| \tag{5.101}$$

Let us write the right hand side of this equation simply as

$$r_{ES}^{(n)} - r_{ES}^{(0)} = \left|\vec{r}_E^{(n)} - \vec{r}_S^{(n)}\right| - \left|\vec{r}_E^{(0)} - \vec{r}_S^{(0)}\right| \tag{5.102}$$

Now, we assume that the distance between the observer and the source is sufficiently larger than the positional variations of the observer or of the source. Namley, we pose a condition as

$$r_{ES}^{(0)} \gg \Delta r_E,\ \Delta r_S \tag{5.103}$$

where

$$r_{ES}^{(0)} \equiv \left|\vec{r}_{ES}^{(0)}\right| \qquad \vec{r}_{ES}^{(0)} \equiv \vec{r}_E^{(0)} - \vec{r}_S^{(0)}, \tag{5.104}$$

$$\Delta r_E \equiv \left|\Delta\vec{r}_E\right|, \qquad \Delta\vec{r}_E \equiv \vec{r}_E^{(n)} - \vec{r}_E^{(0)}, \tag{5.105}$$

$$\Delta r_S \equiv |\Delta \vec{r}_S|, \qquad \Delta \vec{r}_S \equiv \vec{r}_S^{(n)} - \vec{r}_S^{(0)}, \tag{5.106}$$

Then, we can expand Eq.(5.102) as

$$r_{ES}^{(n)} - r_{ES}^{(0)} = \frac{\vec{r}_{ES}^{(0)}}{r_{ES}^{(0)}} \cdot (\Delta \vec{r}_E - \Delta \vec{r}_S) + \cdots \tag{5.107}$$

Therefore the original equation, Eq.(5.101), becomes as

$$t_n = t_0 + nP + \frac{1}{c}\vec{d}_0 \cdot [\Delta \vec{r}_E (t_n) - \Delta \vec{r}_S (t_n)] + \cdots \tag{5.108}$$

where we relabelled the observed timings as

$$t_0 = t_1^{(0)}, \qquad t_n = t_1^{(n)}, \tag{5.109}$$

for simplicity. In the above equation,

$$\vec{d}_0 \equiv \frac{\vec{r}_{ES}^{(0)}}{r_{ES}^{(0)}} \tag{5.110}$$

denotes the direction vector of the source at the epoch of observation t_0,

$$\Delta \vec{r}_E (t_n) \equiv \vec{r}_E (t_n) - \vec{r}_E (t_0) \tag{5.111}$$

denotes the variation of position vector of the observer between at the time t_n and at the epoch t_0, and

$$\Delta \vec{r}_S (t_n) \equiv \vec{r}_S ((t_n - t_0) + T_0) - \vec{r}_S (T_0) \tag{5.112}$$

denotes the variation of position vector of the source between at the epoch of emission $T_0 \equiv t_0^{(0)}$ and at the n-th emission time $t_0^{(n)}$, which is approximated as

$$t_0^{(n)} \approx (t_n - t_0) + T_0 \tag{5.113}$$

Let us introduce a variable

$$\tau_n \equiv t_n - (t_0 + nP) \tag{5.114}$$

This is not the light time we discussed in the previous section but the *deviation* of arrival time from the expected value $t_0 + nP$ for the n-th pulse. We call them the arrival time deviation. By using these, Eq.(5.108) is rewritten as

$$\tau_n = \frac{1}{c}\vec{d}_0 \cdot [\Delta \vec{r}_E (t_0 + nP + \tau_n) - \Delta \vec{r}_S (T_0 + nP + \tau_n)] + \cdots \qquad (n = 0, 1, \cdots) \tag{5.115}$$

This is the equation of pulse time arrival. In principle, there are three kinds of unknowns; the deviation of arrival times, τ_n, the source direction vector at the epoch, $\vec{d}_0$, and the motion[24] of the source at the emission time, $\vec{r}_S(T)$. Thus, it is usually solved by the least square method or other methods handling multiple observation equations simultaneously.

If the source direction and the motion of source are known *a priori* and the higher terms are ignored, however, the equation of pulse time arrival reduces to a single equation

$$f(\tau) \equiv \tau - \frac{1}{c}\vec{d}_0 \cdot \left[\Delta\vec{r}_E\left(t_0 + nP + \tau\right) - \Delta\vec{r}_S\left(T_0 + nP + \tau\right)\right] = 0 \tag{5.116}$$

where the unknown τ_n was rewritten as τ for simplicity. In fact, it is still an equation since τ appears also in the time arguments of terms $\Delta\vec{r}_E$ and $\Delta\vec{r}_S$ in the right hand. The equation is usually solved by the Newton method. The Newton corrector becomes

$$\tau^*\left(\tau\right) \equiv \tau - \frac{f(\tau)}{f'(\tau)} = \frac{\vec{d}_0 \cdot \left[\left(\Delta\vec{r}_{E\tau} - \Delta\vec{r}_{S\tau}\right) - \tau\left(\vec{v}_{E\tau} - \vec{v}_{S\tau}\right)\right]}{c - \vec{d}_0 \cdot \left(\vec{v}_{E\tau} - \vec{v}_{S\tau}\right)} \tag{5.117}$$

where

$$\Delta\vec{r}_{E\tau} \equiv \Delta\vec{r}_E\left(t_0 + nP + \tau\right), \qquad \Delta\vec{r}_{S\tau} \equiv \Delta\vec{r}_S\left(T_0 + nP + \tau\right), \tag{5.118}$$

$$\vec{v}_{E\tau} \equiv \vec{v}_E\left(t_0 + nP + \tau\right), \qquad \vec{v}_{S\tau} \equiv \vec{v}_S\left(T_0 + nP + \tau\right), \tag{5.119}$$

As for the starter, we recommend again the zero starter $\tau_n^{(0)} = 0$. Thus the first application of the Newton method leads to an approximate solution

$$\tau \approx \frac{\vec{d}_0 \cdot \left(\Delta\vec{r}_{En} - \Delta\vec{r}_{Sn}\right)}{c - \vec{d}_0 \cdot \left(\vec{v}_{En} - \vec{v}_{Sn}\right)} \tag{5.120}$$

where

$$\Delta\vec{r}_{En} \equiv \Delta\vec{r}_E\left(t_0 + nP\right), \qquad \Delta\vec{r}_{Sn} \equiv \Delta\vec{r}_S\left(T_0 + nP\right), \tag{5.121}$$

$$\vec{v}_{En} \equiv \vec{v}_E\left(t_0 + nP\right), \qquad \vec{v}_{Sn} \equiv \vec{v}_S\left(T_0 + nP\right), \tag{5.122}$$

[24] This third factor appears only when the motion of source deviates from a straight line motion. Typical cases are when the source is one component of binary or other multiple star systems or when the source has planetary companions. The effect of linear motion is difficult to be detected since it is easily absorbed by the adjustement of epoch of emission, T_0.

Chapter 6

MOTION

6.1 Linear Motions

6.1.1 Purely Linear Motion

Assume a particle, whether target or observer, moves linearly with respect to time. Then, its motion is simply expressed as

$$\vec{r} = \vec{r}_0 + \Delta\vec{r} \tag{6.1}$$

where

$$\Delta\vec{r} = \vec{v}_0(t - t_0) \tag{6.2}$$

is the difference of position vector and t_0 is a certain time of reference, which is called sa the *epoch*. Here

- $\vec{r} \equiv \vec{r}(t)$:

  the position vector of the particle at time t, which is a function of time.

- $\vec{r}_0 \equiv \vec{r}(t_0)$:

  the position vector at the epoch t_0, which is therefore a constant of time.

- $\vec{v}_0 \equiv \mathrm{d}\vec{r}/\mathrm{d}t(t_0)$:

  the velocity vector of the particle at the epoch, which is also a constant of time.

6.1.2 Proper Motion

Frequently, the distance considered is so large that we may assume that the linear motion is applied in polar coordinates

$$\vec{r}(t) = r(t) \begin{pmatrix} \cos\delta(t)\cos\alpha(t) \\ \cos\delta(t)\sin\alpha(t) \\ \sin\delta(t) \end{pmatrix} \tag{6.3}$$

such as

$$r(t) = r_0 + v_r(t - t_0), \qquad \alpha(t) = \alpha_0 + \mu_\alpha(t - t_0), \qquad \delta(t) = \delta_0 + \mu_\delta(t - t_0) \tag{6.4}$$

Here v_r is the radial velocity at the epoch, (α_0, δ_0) is called the mean place[1] at the epoch, and μ_α and μ_δ are called as the proper motion[2] in the right ascension and in the declination, respectively. at the epoch. Occasionally,

$$\mu_{\text{total}} \equiv \sqrt{\mu_\alpha^2 \cos^2\delta + \mu_\delta^2} \tag{6.5}$$

is called as the total proper motion[3] .

6.2 Rotation

6.2.1 Rotation Matrix

One of basic motions of celestial objects is the rotation around their barycenter. Also the rotation play a key role in the transformation of reference frames. Thus, we will discuss some properties of the rotation here. We take a convention that the sense of rotation angle is positive when counter-clockwise and negative when clockwise.

The rotation is generally expressed as an operation multiplying an orthogonal 3×3 matrix[4] to position vectors as

$$\tilde{\vec{r}} = \mathcal{R}\vec{r} \tag{6.8}$$

[1] The word *place* means the place in the celestial sphere, namely the position in two dimensional angular space in this context. In other words, the *place* denotes the direction vector.

[2] The word *proper* means *in angular components* in this context. Sometimes the symbols μ and μ' are used in place of μ_α and μ_δ, respectively. We do not recommend them for they are confusing with other symbols.

[3] Sometimes the symbol μ is used instead. However, we do not recommend its usage again.

[4] Two other schemes have been used for expressing the rotation; *quarternion* and *spinol*. The former was invented by Hamilton as an extension of a complex number. The latter was invented by Cayley and was elaborated by Klein. The spinols are expressed as 2×2 matrix whose components are complex numbers. They have been widely used in quantum mechanics.

The quarternion is expressed as a sort of four vector

$$q = (q_0, q_1, q_2, q_3) = q_0 + q_1\vec{i} + q_2\vec{j} + q_3\vec{k} \tag{6.6}$$

The matrix is called as the rotation matrix. Of course, the rotation matrix corresponding to no rotation is just the unit matrix

$$\mathcal{I} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.9}$$

Remark that the inverse and transpose of the rotation matrices are the same as

$$\mathcal{R}^{-1} = \mathcal{R}^t \tag{6.10}$$

since they are orthogonal. Also note that the determinants of the rotation matrices are always equal to +1

$$\det\mathcal{R} = +1 \tag{6.11}$$

6.2.2 Basic Rotation Matrices

The basic forms of the rotation are the following three one-parameter matrices, which are called as the basic rotation matrices;

$$\mathcal{R}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \tag{6.12}$$

$$\mathcal{R}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{6.13}$$

$$\mathcal{R}_3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.14}$$

Here $\mathcal{R}_j(\theta)$ means the rotation of angle θ around the j-axis. Remark that

$$(\mathcal{R}_j(\theta))^{-1} = \mathcal{R}_j(-\theta) \tag{6.15}$$

---

where the law of multiplication for vector bases $\vec{i}, \vec{j}, \vec{k}$ are defined as

$$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1, \qquad \vec{i}\vec{j} = -\vec{j}\vec{i} = \vec{k} \qquad \vec{j}\vec{k} = -\vec{k}\vec{j} = \vec{i} \qquad \vec{k}\vec{i} = -\vec{i}\vec{k} = \vec{j} \tag{6.7}$$

6.2.3 Euler Angles

A general rotation is a function of three parameters; *rotation angles.* It is expressed as a triple product of the above basic rotation matrices by specifying the indices such as

$$\mathcal{R}_{ijk}\left(\alpha,\beta,\gamma\right) \equiv \mathcal{R}_{k}\left(\gamma\right)\mathcal{R}_{j}\left(\beta\right)\mathcal{R}_{i}\left(\alpha\right) \tag{6.16}$$

Remark that the inverse of a rotation matrix is given by a rotation matrix where the order of indices and arguments and the signature of arguments are reversed as

$$\left(\mathcal{R}_{ijk}\left(\alpha,\beta,\gamma\right)\right)^{-1} = \mathcal{R}_{kji}\left(-\gamma,-\beta,-\alpha\right) \tag{6.17}$$

In principle, any combination is allowed for the indices of the basic rotation matrices. However, the mostly common combination is the following 313-sequence (or x-convention);

$$\mathcal{R}_{313}\left(\phi,\theta,\psi\right) = \mathcal{R}_{3}\left(\psi\right)\mathcal{R}_{1}\left(\theta\right)\mathcal{R}_{3}\left(\phi\right)$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \tag{6.18}$$

In this case, the three rotation angles are called as Euler angles[5] . The rotation specifying the orbital plane by means of Keplerian elements (Ω, I, ω)

$$\mathcal{R} = \mathcal{R}_{3}\left(-\Omega\right)\mathcal{R}_{1}\left(-I\right)\mathcal{R}_{3}\left(-\omega\right) \tag{6.19}$$

is a typical example of 313-sequence.

When all the three angles are small, the 313-sequence matrix is expanded as

$$\mathcal{R}_{313}\left(\phi,\theta,\psi\right) = \mathcal{I} + \Delta_1\mathcal{R}_{313}\left(\phi,\theta,\psi\right) + \Delta_2\mathcal{R}_{313}\left(\phi,\theta,\psi\right) + \cdots \tag{6.20}$$

where $\mathcal{I}$ is a unit matrix, the first order correction is

$$\Delta_1\mathcal{R}_{313}\left(\phi,\theta,\psi\right) = \begin{pmatrix} 0 & \phi+\psi & 0 \\ -(\phi+\psi) & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix}, \tag{6.21}$$

and the second order correction is

$$\Delta_2\mathcal{R}_{313}\left(\phi,\theta,\psi\right) = \frac{-1}{2}\begin{pmatrix} (\phi+\psi)^2 & 0 & -2\psi\theta \\ 0 & (\phi+\psi)^2+\theta^2 & 0 \\ -2\theta\phi & 0 & \theta^2 \end{pmatrix} \tag{6.22}$$

[5] This follows the x- or *continental* convention [Goldstein 1980]. The y- or *British* convention adopts the 323-sequence.

6.2.4 Degenerate Case

Remark that we may rewrite $\mathcal{R}$ as

$$\mathcal{R} = \mathcal{R}_3(\psi)\,\mathcal{R}_1(\theta)\,\mathcal{R}_3(-\psi)\,\mathcal{R}_3(\psi+\phi) = \mathcal{Q}\mathcal{R}_3(\psi+\phi) \tag{6.23}$$

where

$$\mathcal{Q} = \mathcal{R}_{313}(-\psi, \theta, \psi) \tag{6.24}$$

When the second angle θ only is small, the matrix $\mathcal{Q}$ is close to the unit matrix $\mathcal{I}$ as

$$\begin{aligned}\mathcal{Q} &= \begin{pmatrix} \cos^2\psi + \sin^2\psi\cos\theta & -\sin\psi\cos\psi\,(1-\cos\theta) & \sin\psi\sin\theta \\ -\sin\psi\cos\psi\,(1-\cos\theta) & \sin^2\psi + \cos^2\psi\cos\theta & \cos\psi\sin\theta \\ -\sin\psi\sin\theta & -\cos\psi\sin\theta & \cos\theta \end{pmatrix} \\ &= \mathcal{I} - \begin{pmatrix} (1-\cos\theta)\sin^2\psi & (1-\cos\theta)\sin\psi\cos\psi & -\sin\theta\sin\psi \\ (1-\cos\theta)\sin\psi\cos\psi & (1-\cos\theta)\cos^2\psi & -\sin\theta\cos\psi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & (1-\cos\theta) \end{pmatrix}\end{aligned} \tag{6.25}$$

Introduce the variable transformation,

$$p \equiv \tan\frac{\theta}{2}\cos\psi, \qquad q \equiv \tan\frac{\theta}{2}\sin\psi \tag{6.26}$$

Then, by using the formulas

$$1-\cos\theta = \frac{2\tan^2(\theta/2)}{1+\tan^2(\theta/2)}, \qquad \sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} \tag{6.27}$$

$\mathcal{Q}$ is expressed by rational functions of these variables as

$$\mathcal{Q} = \mathcal{I} - \frac{2}{1+p^2+q^2}\begin{pmatrix} q^2 & pq & -q \\ pq & p^2 & -p \\ q & p & p^2+q^2 \end{pmatrix} \tag{6.28}$$

Also, through similar derivations, $\mathcal{Q}$ is expressed in terms of another set of variables

$$\bar{p} \equiv \sin\theta\cos\psi, \qquad \bar{q} \equiv \sin\theta\sin\psi \tag{6.29}$$

as

$$\mathcal{Q} = \begin{pmatrix} 1-\bar{q}^2\bar{r} & -\bar{p}\bar{q}\bar{r} & \bar{q} \\ -\bar{p}\bar{q}\bar{r} & 1-\bar{p}^2\bar{r} & \bar{p} \\ -\bar{q} & -\bar{p} & \bar{c} \end{pmatrix} \tag{6.30}$$

where

$$\bar{r} \equiv \frac{1}{1+\bar{c}}, \qquad \bar{c} \equiv \cos\theta = \sqrt{1-(\bar{p}^2+\bar{q}^2)} \tag{6.31}$$

These forms are used in describing the rotation specifying the orbital planes for the low inclination orbits.

6.2.5 323-Sequence

Consider a rotation around a certain (fixed) direction. If the rotation angle is denoted as χ, and the direction vector is expressed as

$$\vec{n} = \begin{pmatrix} \sin\varphi\cos\lambda \\ \sin\varphi\sin\lambda \\ \cos\varphi \end{pmatrix} \tag{6.32}$$

then the rotation is expressed by a 323-sequence as

$$\mathcal{R} = \mathcal{R}_{323}(\lambda,\varphi,\chi) = \mathcal{R}_3(\chi)\,\mathcal{R}_2(\varphi)\,\mathcal{R}_3(\lambda)$$

$$= \begin{pmatrix} -\sin\chi\sin\lambda + \cos\chi\cos\varphi\cos\lambda & \sin\chi\cos\lambda + \cos\chi\cos\varphi\sin\lambda & -\cos\chi\sin\varphi \\ -\cos\chi\sin\lambda - \sin\chi\cos\varphi\cos\lambda & \cos\chi\cos\lambda - \sin\chi\cos\varphi\sin\lambda & \sin\chi\sin\varphi \\ \sin\varphi\cos\lambda & \sin\varphi\sin\lambda & \cos\varphi \end{pmatrix} \tag{6.33}$$

Note that the 323-sequence is translated to the 313-sequence as

$$\mathcal{R}_3(\chi)\,\mathcal{R}_2(\varphi)\,\mathcal{R}_3(\lambda) = \mathcal{R}_3\left(\chi - \frac{\pi}{2}\right)\mathcal{R}_1(\varphi)\,\mathcal{R}_3\left(\lambda + \frac{\pi}{2}\right) \tag{6.34}$$

Remark that this is also expressed by means of vector products as

$$\tilde{\vec{r}} = \vec{r} + \sin\chi\vec{n}\times\vec{r} + (1-\cos\chi)\,\vec{n}\times(\vec{n}\times\vec{r}) \tag{6.35}$$

where the vector triple product reduces as

$$\vec{n}\times(\vec{n}\times\vec{r}) = (\vec{n}\cdot\vec{r})\,\vec{n} \tag{6.36}$$

The precession matrix

$$\mathcal{P} = \mathcal{R}_3(-z_A)\,\mathcal{R}_2(\theta_A)\,\mathcal{R}_3(-\zeta_A) = \mathcal{R}_{323}(-\zeta_A,\theta_A,-z_A) \tag{6.37}$$

is a typical example of 323-sequence.

When all the three angles are small, the 323-sequence matrix is expanded as

$$\mathcal{R}_{323}(\lambda,\varphi,\chi) = \mathcal{I} + \Delta_1\mathcal{R}_{323}(\lambda,\varphi,\chi) + \Delta_2\mathcal{R}_{323}(\lambda,\varphi,\chi) + \cdots \tag{6.38}$$

where the first order correction is

$$\Delta_1\mathcal{R}_{323}(\lambda,\varphi,\chi) = \begin{pmatrix} 0 & \chi+\lambda & -\varphi \\ -(\chi+\lambda) & 0 & 0 \\ \varphi & 0 & 0 \end{pmatrix} \tag{6.39}$$

and the second order correction is

$$\Delta_2\mathcal{R}_{323}(\lambda,\varphi,\chi) = \frac{-1}{2}\begin{pmatrix} (\chi+\lambda)^2+\varphi^2 & 0 & 0 \\ 0 & (\chi+\lambda)^2 & -2\chi\varphi \\ 0 & -2\varphi\lambda & \varphi^2 \end{pmatrix} \tag{6.40}$$

6.2.6 131-Sequence

The 131-sequence

$$\mathcal{R}_{131}(\rho,\sigma,\tau) = \mathcal{R}_1(\tau)\,\mathcal{R}_3(\sigma)\,\mathcal{R}_1(\rho)$$

$$= \begin{pmatrix} \cos\theta & -\sin\sigma\cos\tau & \sin\sigma\sin\tau \\ \cos\rho\sin\sigma & -\sin\rho\sin\tau + \cos\rho\cos\sigma\cos\tau & -\sin\rho\cos\tau - \cos\rho\cos\sigma\sin\tau \\ \sin\rho\sin\sigma & \cos\rho\sin\tau + \sin\rho\cos\sigma\cos\tau & \cos\rho\cos\tau - \sin\rho\cos\sigma\sin\tau \end{pmatrix} \quad (6.41)$$

also appears in astrometry. It is used in expressing the nutation matrix as

$$\mathcal{N} = \mathcal{R}_1(-(\varepsilon_A + \Delta\varepsilon))\,\mathcal{R}_3(-\Delta\psi)\,\mathcal{R}_1(\varepsilon_A) = \mathcal{R}_{131}(\varepsilon_A, -\Delta\psi, -(\varepsilon_A + \Delta\varepsilon)) \quad (6.42)$$

where ε_A is the mean obliquity, and $\Delta\varepsilon$ and $\Delta\psi$ are nutation in obliquity and in longitude, respectively.

6.2.7 321-Sequence

Neither 313- nor 323-sequence is suitable when the three angles are small so that the rotation matrix is close to the unit matrix. For example, the 313-sequence matrix is expanded as

$$\mathcal{R}_{313}(\psi,\theta,\phi) \approx \begin{pmatrix} 1 & \phi+\psi & 0 \\ -(\phi+\psi) & 1 & \theta \\ 0 & -\theta & 1 \end{pmatrix} \quad (6.43)$$

when $\psi, \theta, \phi \ll 1$. It is easily seen that the angles ϕ and ψ are degenerated, namely only their sum is well-determined, while the rotation around y-axis cannot be represented by the combination of (ϕ, θ, ψ). In such case, we must choose the indices to be different with each other. One typical combination is the 321-sequence

$$\mathcal{R}_{321}(\gamma,\beta,\alpha) = \mathcal{R}_1(\alpha)\,\mathcal{R}_2(\beta)\,\mathcal{R}_3(\gamma)$$

$$= \begin{pmatrix} \cos\beta\cos\gamma & \cos\beta\sin\gamma & -\sin\beta \\ \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\cos\beta \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\cos\beta \end{pmatrix} \quad (6.44)$$

When the angles $\alpha, \beta, \gamma \ll 1$, the matrix is expanded as

$$\mathcal{R}_{321}(\gamma,\beta,\alpha) = \mathcal{I} + \Delta_1\mathcal{R}_{321}(\gamma,\beta,\alpha) + \Delta_2\mathcal{R}_{321}(\gamma,\beta,\alpha) + \cdots \quad (6.45)$$

where the first order correction is

$$\Delta_1 \mathcal{R}_{321}(\gamma, \beta, \alpha) = \begin{pmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0 \end{pmatrix} \tag{6.46}$$

and the second order correction is

$$\Delta_2 \mathcal{R}_{321}(\gamma, \beta, \alpha) = \frac{-1}{2} \begin{pmatrix} \beta^2 + \gamma^2 & 0 & 0 \\ -2\alpha\beta & \gamma^2 + \alpha^2 & 0 \\ -2\gamma\alpha & -2\beta\gamma & \alpha^2 + \beta^2 \end{pmatrix} \tag{6.47}$$

6.2.8 312-Sequence

In astrometry, the 312-sequence is used preferably than the 321-sequence.

$$\mathcal{R}_{312}(\eta, \xi, \zeta) = \mathcal{R}_2(\eta) \mathcal{R}_1(\xi) \mathcal{R}_3(\zeta)$$

$$= \begin{pmatrix} \cos\eta\cos\zeta - \sin\eta\sin\xi\sin\zeta & \cos\eta\sin\zeta + \sin\eta\sin\xi\cos\zeta & -\sin\eta\cos\zeta \\ -\cos\xi\sin\zeta & \cos\xi\cos\zeta & \sin\xi \\ \sin\eta\cos\zeta + \cos\eta\sin\xi\sin\zeta & \sin\eta\sin\zeta - \cos\eta\sin\xi\cos\zeta & \cos\eta\cos\zeta \end{pmatrix} \tag{6.48}$$

The typical example is the combination of polar motion and the siderial rotation of the Earth as

$$\mathcal{WS} = \mathcal{R}_2(-x_p) \mathcal{R}_1(-y_p) \mathcal{R}_3(\Theta). \tag{6.49}$$

When the angles $\xi, \eta, \zeta \ll 1$, the matrix is expanded as

$$\mathcal{R}_{312}(\eta, \xi, \zeta) = \mathcal{I} + \Delta_1 \mathcal{R}_{312}(\eta, \xi, \zeta) + \Delta_2 \mathcal{R}_{312}(\eta, \xi, \zeta) + \cdots \tag{6.50}$$

where the first order correction is

$$\Delta_1 \mathcal{R}_{312}(\eta, \xi, \zeta) = \begin{pmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{pmatrix} \tag{6.51}$$

and the second order correction is

$$\Delta_2 \mathcal{R}_{312}(\eta, \xi, \zeta) = \frac{-1}{2} \begin{pmatrix} \eta^2 + \zeta^2 & -2\eta\xi & 0 \\ 0 & \xi^2 + \zeta^2 & 0 \\ -2\xi\zeta & -2\eta\zeta & \eta^2 + \zeta^2 \end{pmatrix} \tag{6.52}$$

6.2.9 Infinitesimal Rotation

Consider a small rotation such that its second and higher-order effects are negligible. If the rotation is small, the corresponding rotation matrix is written as the sum of the unit matrix plus a matrix whose components are all small as

$$\mathcal{R} = \mathcal{I} + \Delta\mathcal{R} \tag{6.53}$$

By substituting this expression into the rewriting of Eq.(6.10) as

$$\mathcal{R}\mathcal{R}^t = (\mathcal{I} + \Delta\mathcal{R})(\mathcal{I} + \Delta\mathcal{R})^t = \mathcal{I} + \Delta\mathcal{R} + \Delta\mathcal{R}^t + \Delta\mathcal{R}\Delta\mathcal{R}^t = \mathcal{I}, \tag{6.54}$$

and ignoring the higher order term $\Delta\mathcal{R}\Delta\mathcal{R}^t$, we obtain an approximation

$$\Delta\mathcal{R}^t \approx -\Delta\mathcal{R} \tag{6.55}$$

Namely the deviation $\Delta\mathcal{R}$ should be close to an anti-symmetric matrix,

$$\Delta\mathcal{R} \approx \begin{pmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix} \tag{6.56}$$

where θ_j (j=1,2,3) are the three independent components characterizing the inifinitesimal rotation. Remark that a vector product is expressed by means of an anti-symmetric matrix as

$$\vec{\omega} \times \vec{r} = \mathbf{\Omega}\vec{r} \tag{6.57}$$

where

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \text{if} \qquad \vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{6.58}$$

Therefore, the deviation of vector caused by an infinitesimal rotation is equivalent with the vector product;

$$\Delta\mathcal{R}\vec{r} = \vec{\theta} \times \vec{r} + \cdots \tag{6.59}$$

where

$$\vec{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \tag{6.60}$$

If we write the above relation symbolically as

$$\Delta\mathcal{R} \approx \vec{\theta}\times \tag{6.61}$$

then the deviation for a basic rotation matrix becomes

$$\Delta\mathcal{R}_1 \approx \begin{pmatrix} \theta_1 \\ 0 \\ 0 \end{pmatrix} \times, \qquad \Delta\mathcal{R}_2 \approx \begin{pmatrix} 0 \\ \theta_2 \\ 0 \end{pmatrix} \times, \qquad \Delta\mathcal{R}_3 \approx \begin{pmatrix} 0 \\ 0 \\ \theta_3 \end{pmatrix} \times . \tag{6.62}$$

Therefore, the deviation for a general rotation matrix becomes

$$\Delta\mathcal{R}_{ijk}\left(\alpha,\beta,\gamma\right) \approx \Delta\mathcal{R}_i\left(\alpha\right) + \Delta\mathcal{R}_j\left(\beta\right) + \Delta\mathcal{R}_k\left(\gamma\right) \approx \left(\alpha\vec{e}_i + \beta\vec{e}_j + \gamma\vec{e}_k\right) \times . \tag{6.63}$$

6.2.10 Rotational Velocity

The time variation of rotation is expressed as

$$\tilde{\vec{v}} = \frac{\mathrm{d}\mathcal{R}}{\mathrm{dt}}\vec{r} + \mathcal{R}\vec{v} \tag{6.64}$$

Here the time derivative of the rotation matrix is given as

$$\frac{\mathrm{d}\mathcal{R}_{ijk}\left(\theta_i,\theta_j,\theta_k\right)}{\mathrm{dt}} = \frac{\mathrm{d}\mathcal{R}_k\left(\theta_k\right)}{\mathrm{dt}}\mathcal{R}_j\left(\theta_j\right)\mathcal{R}_i\left(\theta_i\right)$$

$$+\mathcal{R}_k\left(\theta_k\right)\frac{\mathrm{d}\mathcal{R}_j\left(\theta_j\right)}{\mathrm{dt}}\mathcal{R}_i\left(\theta_i\right) + \mathcal{R}_k\left(\theta_k\right)\mathcal{R}_j\left(\theta_j\right)\frac{\mathrm{d}\mathcal{R}_i\left(\theta_i\right)}{\mathrm{dt}} \tag{6.65}$$

where the time derivatives of the basic rotation matrices are

$$\frac{\mathrm{d}\mathcal{R}_j\left(\theta\right)}{\mathrm{dt}} = \left(\frac{\mathrm{d}\mathcal{R}_j\left(\theta\right)}{\mathrm{d}\theta}\right)\frac{\mathrm{d}\theta}{\mathrm{dt}} = \mathcal{Q}_j\left(\theta\right)\frac{\mathrm{d}\theta}{\mathrm{dt}} \tag{6.66}$$

and

$$\mathcal{Q}_1\left(\theta\right) \equiv \frac{\mathrm{d}\mathcal{R}_1\left(\theta\right)}{\mathrm{d}\theta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \\ 0 & -\cos\theta & -\sin\theta \end{pmatrix}, \tag{6.67}$$

$$\mathcal{Q}_2\left(\theta\right) \equiv \frac{\mathrm{d}\mathcal{R}_2\left(\theta\right)}{\mathrm{d}\theta} = \begin{pmatrix} -\sin\theta & 0 & -\cos\theta \\ 0 & 0 & 0 \\ \cos\theta & 0 & -\sin\theta \end{pmatrix}, \tag{6.68}$$

$$\mathcal{Q}_3\left(\theta\right) \equiv \frac{\mathrm{d}\mathcal{R}_3\left(\theta\right)}{\mathrm{d}\theta} = \begin{pmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.69}$$

6.3 Keplerian Motions

The most important motion dealt in the fundamental astronomy is the Keplerian motion, which is the exact solution of two body problems under the Newton's universal attraction. The Keplerian motion is characterized by a set of six constants named the Keplerian elements[6] , and one parameter called the gravitational constant[7] . The gravitational constant is denoted by μ and appears in the equation of motion as

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \frac{-\mu}{r^3}\vec{r} \tag{6.70}$$

where $\vec{r}$ is the position vector relative to the gravitating body.

6.3.1 Keplerian Elements

There are variety of choices of Keplerian elements. One of the most popular sets is $(a, e, I, \Omega, \omega, T)$[8] . Here

- a: the semi-major axis[9] ; $(0 < a)$

  the half of the length of the longer axis of ellipse[10]

---

[6] Also called as *orbital elements* frequently.

[7] It is frequently attached with the adjective specifying the gravitating body such as *heliocentric* gravitational constant GM_{Sun} as that of the Sun, or as *geocentric* gravitational constant GM_{Earth} as that of the Earth. This is in order to discriminate them from the constant of universal attraction G.

[8] Popular variations of Keplerian elements in the elliptic case are

1. Usage of the mean anomaly at a specific epoch t_0, $M_0 \equiv n(t_0 - T)$, instead of T.
2. Usage of the pericenter distance, $q \equiv a(1-e)$, instead of a.
3. Usage of the mean motion, $n \equiv \sqrt{\mu/a^3}$, instead of a.
4. Usage of the orbital period, $P \equiv 2\pi/n$, instead of a.
5. Usage of the longitude of pericenter, $\varpi \equiv \omega + \Omega$, instead of ω.
6. Usage of the mean orbital longitude at the epoch, $L_0 \equiv M_0 + \omega + \Omega$, instead of M_0.
7. Usage of the pair $(\sin I \cos\Omega, \sin I \sin\Omega)$, instead of the pair (I, Ω).
8. Usage of the pair $(\tan(I/2)\cos\Omega, \tan(I/2)\sin\Omega)$, instead of the pair (I, Ω).
9. Usage of the pair $(e\cos\omega, e\sin\omega)$, instead of the pair (e, ω).
10. Usage of the pair $(e\cos\varpi, e\sin\varpi)$, instead of the pair (e, ϖ).

[9] The word *semi* means *the half of* in this context. The *major axis* means the longer axis of an ellipse.

[10] In the elliptic case.

- e: the eccentricity; $(0 \leq e)$

  the ratio of the distance between the center and a focus of the ellipse to the semi-major axis[11] .

- I: the inclination[12] ; $(0 \leq I < \pi)$

  the angle between the orbital plane and the x-y plane of the reference frame.

- Ω: the longitude of ascending node[13] ; $(0 \leq \Omega < 2\pi)$

  the angle between the x-axis and the ascending node in the x-y plane of the reference frame.

- ω: the argument of pericenter[14] ; $(0 \leq \omega < 2\pi)$

  the angle between the ascending node and the pericenter in the orbital plane.

- T: the time of pericenter passage[15]

These are categorized into the following three;

1. the pair (a, e) characterizing the shape of the orbit,
2. the quantity T specifying a fiducial point on the orbit, and
3. the trio (Ω, I, ω) determining the orientation of the orbital plane.

According to the value of e, the shape of the orbit differs as

1. an ellipse when $e < 1$,
2. a parabola when $e = 1$, and
3. a hyperbola when $e > 1$

[11] In the elliptic case.

[12] Sometimes, a small letter i is used instead. We prefer I since i is confusing with $\sqrt{-1}$

[13] The word *longitude* means the angle measured from the fiducial point, the x-axis of the reference frame. The word *node* means the nodal points defined as two intersections of the nodal line to the unit circle on the x-y plane. Here the nodal line is the intersection between the orbital plane and the x-y plane of the reference frame. There are two nodal points. The ascending node is the one across which the orbitting body passes from the southern hemisphere to the northern one. Here the northern (southern) hemisphere is the region where $z > 0$ ($z < 0$).

[14] The word *argument* means the angle measured from the ascending node in this context. The word *peri-* means being the closest. The *center* denotes not the center of the ellipse but the gravitating body.

[15] Sometimes the symbol t_0 is used instead. We prefer T since we want to reserve t_0 for epochs.

Table 6.1: Angles Used in Orbital Motions

Symbol	Name	Definition
Anomaly		
f	True Anomaly	$f \equiv \tan^{-1}(\eta/\xi)$
M (or ℓ)	Mean Anomaly	$M \equiv n(t-T)$
E (or u)	Eccentric Anomaly	$E - e\sin E = M$
Argument		
ω	Argument of Pericenter	
v	Argument of Latitude	$v \equiv f + \omega$
N	Mean Argument of Latitude	$N \equiv M + \omega$
D	Eccentric Argument of Latitude	$D \equiv E + \omega$
Longitude		
Ω	Longitude of Ascending Node	
ϖ	Longitude of Pericenter	$\varpi \equiv \omega + \Omega$
λ	True Longitude	$\lambda \equiv f + \omega + \Omega$
L	Mean Longitude	$L \equiv M + \omega + \Omega$
Λ	Eccentric Longitude	$\Lambda \equiv E + \omega + \Omega$

Note: *Anomaly* means the angle measured from the pericenter. *Argument* means the angle measured from the ascending node. *Longitude* means the angle measured from the fiducial point, namely the x-axis of the frame of reference.

6.3.2 Elliptic Orbit

Here we will describe how to obtain the three dimensional position and velocity as a function of time t and the given Keplerian elements of an object orbitting a Keplerian motion.

First, consider the time variation of two-dimensional coordinates on the orbital plane. The Keplerian motion is classified into three cases; elliptic[16] , parabolic, and hyperbolic [Danby 1988].

Let us consider the elliptic case. Take a coordinate system such that the gravitating body locates at the origin and the pericenter is on the ξ-axis. Here the pericenter is, in the orbit, the closest point to the gravitating body. Then the position vector in the orbital plane (ξ, η) are described in terms of a, e, and a variable E named the eccentric anomaly[17] as

$$\xi = a\left(\cos E - e\right), \qquad \eta = b \sin E \tag{6.71}$$

where

$$b = a e' \tag{6.72}$$

is the semi-minor axis[18] , and

$$e' = \sqrt{1 - e^2} \tag{6.73}$$

is the complementary eccentricity. Remark that a, b, and e are constants of time. While the eccentric anomaly is usually a complicated function of time. It is obtained by solving the following nonlinear equation named as Kepler's equation [Colwell 1993]

$$E - e \sin E = M \tag{6.74}$$

Here M is a linear function of time t defined as

$$M = n\left(t - T\right) \tag{6.75}$$

and is called as the mean anomaly[19] . And n is a constant derived from μ and a by Kepler's third law[20] as

$$n = \sqrt{\frac{\mu}{a^3}} \tag{6.76}$$

[16] The circular case is usually included into the elliptic one, although some quantities will be indeterminate in the limit of circular orbit.

[17] The *anomaly* is the angle measured from the pericenter. Originally *anomaly* is the abbrevieation of the anomalistic angle, namely the angle deviation.

[18] The *minor axis* means the shorter axis of the ellipse.

[19] Sometimes the pair of symbols (ℓ, u) is used instead of (M, E). In that case, (the elliptic) Kepler"s equation is written as $u - e \sin u = \ell$

[20] It is best remembered as $\mu = n^2 a^3$.

and is called as the mean motion. The question how to solve Kepler's equation will be discussed later.

Remark that the specific[21] energy $\mathcal{E}$ and the specific angular momentum $\mathcal{L}$ are given as

$$\mathcal{E} \equiv \frac{\vec{v}^2}{2} + \frac{\mu}{|\vec{r}|} = \frac{-\mu}{2a} < 0, \qquad \mathcal{L} \equiv |\vec{r} \times \vec{v}| = \sqrt{\mu a(1-e^2)} \tag{6.77}$$

On the other hand, the velocity components on the orbital plane are simply obtained by differentiating Eq.(6.71) with respect to time as

$$\frac{\mathrm{d}\xi}{\mathrm{dt}} = -a \sin E \frac{\mathrm{d}E}{\mathrm{dt}}, \qquad \frac{\mathrm{d}\eta}{\mathrm{dt}} = b \cos E \frac{\mathrm{d}E}{\mathrm{dt}} \tag{6.78}$$

where the time derivative of the eccentric anomaly is given as

$$\frac{\mathrm{d}E}{\mathrm{dt}} = \frac{n}{1 - e \cos E} \tag{6.79}$$

which is easily derived by differentiating Eq.(6.74). Remark that the radius vector r is calculated as

$$r \equiv \sqrt{\xi^2 + \eta^2} = a(1 - e \cos E). \tag{6.80}$$

Therefore, the pericenter and apocenter[22] distances are obtained as

$$q \equiv \min_E r = r_{E=0} = a(1-e), \qquad Q \equiv \max_E r = r_{E=\pi} = a(1+e). \tag{6.81}$$

On the other hand the magnitude of velocity v is computed as

$$v \equiv \sqrt{\left(\frac{\mathrm{d}\xi}{\mathrm{dt}}\right)^2 + \left(\frac{\mathrm{d}\eta}{\mathrm{dt}}\right)^2} = na\sqrt{\frac{1 + e \cos E}{1 - e \cos E}} \tag{6.82}$$

Therefore, the pericenter and apocenter velocities are expressed as

$$v_{\mathrm{Pericenter}} = na\sqrt{\frac{1+e}{1-e}} = \max_E v, \qquad v_{\mathrm{Apocenter}} = na\sqrt{\frac{1-e}{1+e}} = \min_E v, \tag{6.83}$$

[21] The word *specific* means quantities *per unit mass.*
[22] The word *apo-* means being the farthest.

Table 6.2: Formulas of Keplerian orbits: elliptic case

Item	Formula
Parameters	
Eccentricity	$0 \leq e < 1$
Semi-Major Axis	a
Constants	
Compl. Eccentricity	$e' = \sqrt{1-e^2}$
Semi-Minor Axis	$b = ae'$
Energy	$\mathcal{E} = -\mu/(2a) < 0$
Angular Momentum	$\mathcal{L} = \sqrt{\mu a(1-e^2)}$
Mean Motion	$n = \sqrt{\mu/a^3}$
Kepler's Equation	
Eccentric Anomaly	E
Mean Anomaly	$M = n(t-T)$
Equation	$E - e\sin E = M$
Newton Corrector	$E^*(E) \equiv [M + e(\sin E - E\cos E)]/(1 - e\cos E)$
Starter	$S = \min\left(E^*(0), E^*(\pi/2), E^*(\pi)\right)$
	$E^*(0) = M/(1-e)$
	$E^*(\pi/2) = M + e$
	$E^*(\pi) = (M + \pi e)/(1+e)$
Newton Iteration	$E \leftarrow E^*(E)$
Position	
x-component	$\xi = a(\cos E - e)$
y-component	$\eta = b\sin E$
Distance	
Radius vector	$r = a(1 - e\cos E)$
Pericenter	$q = a(1-e)$
Apocenter	$Q = a(1+e)$
Time Derivative of	
Eccentric Anomaly	$\dot{E} = na/(1 - e\cos E)$
Velocity	
x-component	$\dot{\xi} = -a\dot{E}\sin E$
y-component	$\dot{\eta} = b\dot{E}\cos E$
Total	$v = a\dot{E}\sqrt{1 - e^2\cos^2 E}$

Note: μ is the gravitational constant of the Keplerian motion.

6.3.3 Parabolic Orbit

As is seen from the fator $\sqrt{1-e^2}$ in Eq.(6.71), the formulas in the previous section are inappropriate when $e \geq 1$. The case $e = 1$ is called to be parabolic. In that case, the position vector in the orbital plane is given as

$$\xi = q\left(1-\tau^2\right), \qquad \eta = 2q\tau \tag{6.84}$$

Here the quantitity τ, which may be called as the eccentric anomaly of the parabolic orbit, is defined by means of the true anomaly[23] f as

$$\tau \equiv \tan\frac{f}{2} \tag{6.85}$$

Remark that f is the angle in the polar coordinate such as

$$\xi = r\cos f, \qquad \eta = r\sin f \tag{6.86}$$

The quantity τ is obtained by solving the parabolic version of Kepler's equation, named Barker's equation[24] ,

$$\tau + \frac{\tau^3}{3} = M_P \tag{6.91}$$

[23] Sometimes the symbol v is used instead. We prefer the symbol f since we keep v for expressing the magnitude of velocity.

[24] Barker's equation, Eq.(6.91), is derived from the (elliptic) Kepler's equation in a straightforward manner. By putting

$$a = \frac{q}{1-e} \tag{6.87}$$

in the expression of M and expanding $\sin E$ with respect to E, we rewrote the elliptic Kepler's equation, Eq.(6.74), as

$$E - e\sin E = (1-e)E + e\frac{E^3}{6}\left(1 - \frac{E^2}{20} + \frac{E^4}{840} - \cdots\right) = \sqrt{1-e}^{\,3}\sqrt{\frac{\mu}{q^3}}(t-T) \tag{6.88}$$

Changing the variable from E to τ as

$$E = \tau\sqrt{2(1-e)} \tag{6.89}$$

we obtain

$$\tau + e\frac{\tau^3}{3}\left(1 - (1-e)\frac{\tau^2}{20} + (1-e)^2\frac{\tau^4}{210} - \cdots\right) = \sqrt{\frac{\mu}{2q^3}}(t-T) = M_P \tag{6.90}$$

Then, by taking the limit $e \to 1$ in this rewriting, we obtain Barker's equation.

Here M_P is the parabolic mean aomaly defined as

$$M_P = n_P(t - T) \tag{6.92}$$

where

$$n_P = \sqrt{\frac{\mu}{2q^3}} \tag{6.93}$$

is the parabolic mean motion. The velocity vector is given as

$$\frac{\mathrm{d}\xi}{\mathrm{dt}} = -2q\tau\frac{\mathrm{d}\tau}{\mathrm{dt}}, \qquad \frac{\mathrm{d}\eta}{\mathrm{dt}} = 2q\frac{\mathrm{d}\tau}{\mathrm{dt}} \tag{6.94}$$

where the time derivative of τ is expressed as

$$\frac{\mathrm{d}\tau}{\mathrm{dt}} = \frac{n_P}{1+\tau^2} \tag{6.95}$$

Remark that the specific energy $\mathcal{E}$ and the specific angular momentum $\mathcal{L}$ are given as

$$\mathcal{E} = 0, \qquad \mathcal{L} = \sqrt{2\mu q} \tag{6.96}$$

Remark that the radius vector r is calculated as

$$r = q\left(1+\tau^2\right). \tag{6.97}$$

On the other hand the magnitude of velocity v is computed as

$$v = \frac{2qn_P}{\sqrt{1+\tau^2}} \tag{6.98}$$

Therefore, the pericenter velocity is expressed as

$$v_{\mathrm{Pericenter}} = 2qn_P = \max_E v, \tag{6.99}$$

6.3.4 Hyperbolic Orbit

In the case $e > 1$, the position in the orbital plane is given as

$$\xi = a_H\left(e - \cosh F\right), \qquad \eta = b_H \sinh F \tag{6.100}$$

Here a_H is the hyperbolic semi-major axis and

$$b_H = a_H\sqrt{e^2-1} \tag{6.101}$$

Table 6.3: Formulas of Keplerian orbits: parabolic case

Item	Formula
Parameters	
Eccentricity	$e = 1$
Pericenter Distance	q
Constants	
Energy	$\mathcal{E} = 0$
Angular Momentum	$\mathcal{L} = \sqrt{2\mu q}$
Mean Motion	$n_P = \sqrt{\mu/(2q^3)}$
Kepler's Equation	
Eccentric Anomaly	$\tau \equiv \tan(f/2)$
Mean Anomaly	$M_P = n_P(t - T)$
Equation	$\tau + \tau^3/3 = M_P$
Newton Corrector	$\tau^*(\tau) \equiv \left[M_P + (2/3)\tau^3\right] / \left(1 + \tau^2\right)$
Starter	$S_P = \min\left(\tau^*(0), \tau^*(1), \tau^*(7)\right)$
	$\tau^*(0) = M_P$
	$\tau^*(1) = (1/2)M_P + (1/3)$
	$\tau^*(7) = (1/50)M_P + (343/75)$
Newton Iteration	$\tau \leftarrow \tau^*(\tau)$
Position	
x-component	$\xi = q(1 - \tau^2)$
y-component	$\eta = 2q\tau$
Distance	
Radius vector	$r = q(1 + \tau^2)$
Time Derivative of	
Eccentric Anomaly	$\dot{\tau} = n_P/(1 + \tau^2)$
Velocity	
x-component	$\dot{\xi} = -2q\dot{\tau}\tau$
y-component	$\dot{\eta} = 2q\dot{\tau}$
Total	$v = 2q\dot{\tau}\sqrt{1 + \tau^2}$

Note: f is the true anomaly defined as $f \equiv \tan^{-1}(\eta/\xi)$.

is the hyperbolic semi-minor axis. And F is the hyperbolic eccentric anomaly, which is obtained by solving the hyperbolic Kepler's equation[25]

$$e \sinh F - F = M_H \tag{6.103}$$

where M_H is the hyperbolic mean anomaly defined as

$$M_H = n_H \left(t - T\right) \tag{6.104}$$

where n_H is the hyperbolic mean motion defined as

$$n_H = \sqrt{\frac{\mu}{a_H^3}} \tag{6.105}$$

The velocity is given as

$$\frac{\mathrm{d}\xi}{\mathrm{dt}} = -a_H \sinh F \frac{\mathrm{d}F}{\mathrm{dt}}, \qquad \frac{\mathrm{d}\eta}{\mathrm{dt}} = b_H \cosh F \frac{\mathrm{d}F}{\mathrm{dt}} \tag{6.106}$$

where

$$\frac{\mathrm{d}F}{\mathrm{dt}} = \frac{n_H}{e \cosh F - 1} \tag{6.107}$$

Remark that the specific energy $\mathcal{E}$ and the specific angular momentum $\mathcal{L}$ are given as

$$\mathcal{E} = \frac{\mu}{2a_H} > 0, \qquad \mathcal{L} = \sqrt{\mu a_H (e^2 - 1)} \tag{6.108}$$

Remark that the radius vector r is calculated as

$$r = a_H \left(e \cosh F - 1\right). \tag{6.109}$$

On the other hand the magnitude of velocity v is computed as

$$v = n_H a_H \sqrt{\frac{e \cosh F + 1}{e \cosh F - 1}} \tag{6.110}$$

Therefore, the pericenter velocity is expressed as

$$v_{\mathrm{Pericenter}} = n_H a_H \sqrt{\frac{e+1}{e-1}} = \max_E v, \tag{6.111}$$

[25] The hyperbolic quantities are derived from the elliptic ones by the following transformation formulas;

$$a \to -a_H, \qquad E \to F\sqrt{-1}, \qquad M \to -M_H\sqrt{-1} \tag{6.102}$$

Table 6.4: Formulas of Keplerian orbits: hyperbolic case

Item	Formula
Parameters	
Eccentricity	$e > 1$
Semi-Major Axis	a_H
Constants	
Compl. Eccentricity	$e'_H = \sqrt{e^2 - 1}$
Semi-Minor Axis	$b_H = a_H e'_H$
Energy	$\mathcal{E} = \mu/(2a_H) > 0$
Angular Momentum	$\mathcal{L} = \sqrt{\mu a_H (e^2 - 1)}$
Mean Motion	$n_H = \sqrt{\mu/a_H^3}$
Kepler's Equation	
Eccentric Anomaly	F
Mean Anomaly	$M_H = n_H(t - T)$
Equation	$e \sinh F - F = M_H$
Newton Corrector	$F^*(F) \equiv [M_H + e(F \cosh F - \sinh F)] \,/\, (e \cosh F - 1)$
Starter	$S_H = \min\left(F^*(0), F^*(\ln 2), F^*(8)\right)$
	$F*(0) = M_H/(e-1)$
	$F*(\ln 2) = \left[M_H + (0.75 - 0.25 \ln 2) e\right] / (1.25e - 1)$
	$F*(8) = \left[M_H + (8 \cosh 8 - \sinh 8) e\right] / [(\cosh 8) e - 1]$
Newton Iteration	$F \leftarrow F^*(F)$
Position	
x-component	$\xi = a_H(e - \cosh F)$
y-component	$\eta = b_H \sinh F$
Distance	
Radius vector	$r = a_H(e \cosh F - 1)$
Pericenter	$q = a_H(e - 1)$
Time Derivative of	
Eccentric Anomaly	$\dot{F} = n_H a_H/(e \cos F - 1)$
Velocity	
x-component	$\dot{\xi} = -a_H \dot{F} \sinh F$
y-component	$\dot{\eta} = b_H \dot{F} \cosh F$
Total	$v = a_H \dot{F} \sqrt{e^2 \cosh^2 F - 1}$

Note: μ is the gravitational constant of the Keplerian motion.

6.3.5 Orientation

Now the position and velocity vectors in the orbital plane are obtained for the elliptic, the parabolic, and the hyperbolic case. See the summary in Tables 6.2 and 6.3. Next, we consider the orientation of the orbital plane. The coordinate transformation is realized by the 313-sequence as

$$\begin{pmatrix} \vec{r} & \vec{v} \end{pmatrix} = \mathcal{R}_3\left(-\Omega\right)\mathcal{R}_1\left(-I\right)\mathcal{R}_3\left(-\omega\right)\begin{pmatrix} \xi & \mathrm{d}\xi/\mathrm{dt} \\ \eta & \mathrm{d}\eta/\mathrm{dt} \\ 0 & 0 \end{pmatrix}, \tag{6.112}$$

Remark that the direction of the angular momentum vector is given as

$$\frac{\vec{L}}{|\vec{L}|} = \begin{pmatrix} -\sin I \sin\Omega \\ \sin I \cos\Omega \\ \cos I \end{pmatrix} \tag{6.113}$$

6.3.6 Determination of Elements

Now, let us consider the inverse procedure; how to compute the Keplerian elements from the position, $\vec{r}$, and the velocity, $\vec{v}$, at the time given, t.

First, we evaluate the specific energy as

$$\mathcal{E} = \frac{\vec{v}^2}{2} + \frac{\mu}{|\vec{r}|} \tag{6.114}$$

Depending on its signature, the orbit is classified into three cases; elliptic, parabolic, and hyperbolic. Namely, if $\mathcal{E} < 0$, then the orbit is elliptic and

$$a = \frac{-\mu}{2\mathcal{E}}. \tag{6.115}$$

Else if $\mathcal{E} > 0$, then the orbit is hyperbolic and

$$a_H = \frac{\mu}{2\mathcal{E}}. \tag{6.116}$$

Else, then the orbit is parabolic.

Next, we compute the specific angular momentum vector as

$$\vec{L} \equiv \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \vec{r} \times \vec{v} \tag{6.117}$$

Then the angles I and Ω are obtained from Eq.(6.113) as

$$\Omega = \mathrm{atan2}\left(-L_x, L_y\right), \qquad I = \mathrm{atan2}\left(\sqrt{L_x^2 + L_y^2}, L_z\right) \tag{6.118}$$

On the other hand, the direction of pericenter is obtained as another constant vector named as the Laplace vector[26]

$$\vec{e} = \frac{\mathcal{L}}{\mu}\vec{v} + \frac{\vec{r} \times \vec{L}}{r\mathcal{L}} \tag{6.119}$$

where

$$\mathcal{L} \equiv |\vec{L}| = \sqrt{L_x^2 + L_y^2 + L_z^2} \tag{6.120}$$

Then, the argument of pericenter is obtained from the Laplace vector as

$$\omega = \mathrm{atan2}(e_Y, e_X) \tag{6.121}$$

where

$$\begin{pmatrix} e_X \\ e_Y \\ e_Z \end{pmatrix} = \mathcal{R}_1(I)\mathcal{R}_3(\Omega)\vec{e} \tag{6.122}$$

Remark that $e_Z = 0$, which can be used as a numerical check.

As for the eccentricity e, one way is to use the magnitude of the Laplace vector as

$$e = |\vec{e}| \tag{6.123}$$

Another, which is efficient when e is close to one, is to use a formula using the total angular momentum $\mathcal{L}$ and the energy $\mathcal{E}$ as

$$e = \sqrt{1 + \frac{2\mathcal{L}\mathcal{E}}{\mu^2}} \tag{6.124}$$

Now that the three Euler angles of the orbital plane are solved, the position vector on the orbital plane is obtained as

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \mathcal{R}_3(\omega)\mathcal{R}_1(I)\mathcal{R}_3(\Omega)\vec{r} \tag{6.125}$$

where the fact $\zeta = 0$ can be used as a numerical check.

The part to obtain T depends on the value of e. First, consider the elliptic case, namely when $\mathcal{E} < 0$. The eccentric anomaly at t is obtained as

$$E = \mathrm{atan2}(\eta, e'(\xi + ae)) \tag{6.126}$$

[26] The magnitude is equal to the eccentricity, e, and the direction is toward the pericenter. The Laplace vector is also known as the Runge-Lentz vector in electrodynamics.

where

$$e' \equiv \sqrt{1-e^2} = \sqrt{\frac{-2\mathcal{L}\mathcal{E}}{\mu^2}} \tag{6.127}$$

Then the mean anomaly at t is computed by Kepler's equation as

$$M = E - e \sin E \tag{6.128}$$

Thus, the time of pericenter passage T is obtained as

$$T = t - \frac{M}{n} \tag{6.129}$$

where

$$n = \sqrt{\frac{\mu}{a^3}} \tag{6.130}$$

Next, consider the hyperbolic case, when $\mathcal{E} > 0$. The hyperbolic eccentric anomaly at t is obtained as

$$F = \tanh^{-1} \frac{\eta}{e'_H \left(a_H e - \xi\right)} \tag{6.131}$$

where

$$e'_H \equiv \sqrt{e^2 - 1} = \sqrt{\frac{2\mathcal{L}\mathcal{E}}{\mu^2}} \tag{6.132}$$

Then the hyperbolic mean anomaly at t is computed by Kepler's equation as

$$M_H = e \sinh F - F \tag{6.133}$$

Thus, the time of pericenter passage T is obtained as

$$T = t - \frac{M_H}{n_H} \tag{6.134}$$

where

$$n_H = \sqrt{\frac{\mu}{a_H^3}} \tag{6.135}$$

Finally, consider the parabolic case, namely when $\mathcal{E} = 0$, although this case is rarely met in the actual computation. The true anomaly is obtained as

$$f = \text{atan2}(\eta, \xi) \tag{6.136}$$

Then the parabolic eccentric anomaly at t is obtained as

$$\tau = \tan \frac{f}{2} \tag{6.137}$$

Thus the parabolic mean anomaly at t is computed by Kepler's equation as

$$M_P = \tau + \frac{\tau^3}{3} \tag{6.138}$$

Thus, the time of pericenter passage T is obtained as

$$T = t - \frac{M_P}{n_P} \tag{6.139}$$

where

$$n_P = \sqrt{\frac{\mu}{2q^3}} \tag{6.140}$$

while the pericenter distance q is obtained from the total angular momentum $\mathcal{L}$ as

$$q = \frac{\mathcal{L}^2}{2\mu} \tag{6.141}$$

6.4 Perturbed Keplerian Orbits

Most of the actual orbital motions are approximated as pure Keplerian motions. Typical examples are asteroids, comets, binary stars, and other objects with less precise observational data.

On the other hand, the orbital motions of some objects, the major planets, satellites, and rings, are described by a modification of Keplerian elements. Its typical expression is to give Keplerian elements not as a set of constants but as a set of functions of time as

$$\vec{E} = \vec{E}(t) \tag{6.142}$$

where $\vec{E}$ denotes the set of Keplerian orbits like

$$\vec{E}^t = (a, e, I, \Omega, \omega, L) \tag{6.143}$$

Such motion is called the perturbed Keplerian motion. In this case, the Keplerian elements are called to be *osculating*[27] . Generally, the osculating elements are expressed as the sum of a low order polynomial of time and Fourier series of some specific frequencies as

$$\vec{E} = \vec{E}_0 + \vec{E}_1(t - t_0) + \vec{E}_2(t - t_0)^2 + \cdots$$

[27] The word *osculating* means contacting

$$+\sum_j \left(\vec{C}_j \cos\nu_j(t-t_0) + \vec{S}_j \sin\nu_j(t-t_0)\right) \tag{6.144}$$

where the frequencies are usually a linear combination of some basic ones as

$$\nu_j = \sum_{k=1}^{K} n_j^{(k)} \omega_k \tag{6.145}$$

Examples of seular variations are shown for the planets.

On the other hands, the typical periodic variations take the following forms;

$$\Delta e = \sum_j C_j \cos M_j, \qquad e\Delta\varpi = \sum_j S_j \sin M_j, \tag{6.146}$$

or

$$\Delta I = \sum_j I_j^{(p)} \cos\Omega_j^{(p)}, \qquad (\sin I)\Delta\Omega = \sum_j I_j^{(p)} \sin\Omega_j^{(p)},$$

$$\Delta\varpi = \Delta L = \tan\frac{I}{2}\sum_j I_j^{(p)} \sin\Omega_j^{(p)} \tag{6.147}$$

or

$$\Delta(e\cos\varpi) = \sum_j e_j^{(p)} \cos\varpi_j^{(p)}, \qquad \Delta(e\sin\varpi) = \sum_j e_j^{(p)} \sin\varpi_j^{(p)}, \tag{6.148}$$

where C_j denotes some constants, the super script (p) means the proper quantities[28] . Examples of these are shown for some satellites.

[28] The word *proper* means that the quantities referred not to the reference frame but to the Laplacian plane, i.e. the invariable plane, in this context.

Table 6.5: Secular Variations of the Keplerian Elements of Inner Planets

Planet	Element	Constant	Linear Coeff.
Mercury	a	0.38710	1
	e	0.20563	25
	I	7.005	-24
	Ω	48.332	-446
	ω	77.456	574
	L	252.251	538101629
Venus	a	0.72333	1
	e	0.00677	-49
	I	3.395	-3
	Ω	76.681	-997
	ω	131.533	-109
	L	181.980	210664136
Earth+Moon	a	1.00000	0
	e	0.01671	-38
	I	0	-47
	ϖ	91.687	-17030
	L	100.464	129597741
Mars	a	1.52366	-72
	e	0.09341	119
	I	1.851	-25
	Ω	49.579	-1020
	ω	336.041	1561
	L	355.453	68905104

Note: Taken from Table 5.8.1 of [Seidelmann (ed.) 1992]. The epoch is J2000.0 (= JD 2451545.0). The units for constants are AU for a, degrees for the angles. While those for linear coefficients are 10^{-6} AU/jc for a, 10^{-6}/jc for e, and "/jc for the angles. The reference frame is the mean ecliptic and equinox of J2000.0. As a result, the inclination of the Earth+Moon is 0 by definition. Therefore, we listed ϖ only instead of the pair of Ω and ω in this case.

Table 6.6: Secular Variations of the Keplerian Elements of Outer Planets

Planet	Element	Constant	Linear Coeff.
Jupiter	a	5.20336	607
	e	0.04839	−129
	I	1.305	−4
	Ω	100.556	1217
	ω	14.754	840
	L	34.404	10925078
Saturn	a	9.53707	−3015
	e	0.05415	−368
	I	2.484	6
	Ω	113.715	−1591
	ω	92.432	−1949
	L	49.944	4401053
Uranus	a	19.1913	1520
	e	0.04717	−192
	I	0.770	−2
	Ω	74.230	1681
	ω	170.964	1313
	L	313.232	1542548
Neptune	a	30.0690	−1252
	e	0.00859	25
	I	1.769	−4
	Ω	131.722	−151
	ω	44.971	−844
	L	304.880	786449
Pluto+Charon	a	39.4817	−769
	e	0.24881	65
	I	17.142	11
	Ω	110.303	−37
	ω	224.067	−132
	L	238.929	522748

Note: Taken from Table 5.8.1 of [Seidelmann (ed.) 1992]. The epoch is J2000.0 (= JD 2451545.0). The elements for Jupiter, Saturn, Uranus, and Neptune are those for the barycenter of the planet systems, such as the Jovian system for Jupiter. The units for constants are AU for a, degrees for the angles. While those for linear coefficients are 10^{-6} AU/jc for a, 10^{-6}/jc for e, and "/jc for the angles. The reference frame is the mean ecliptic and equinox of J2000.0.

Chapter 7

ADVANCED TOPICS

7.1 Earth Rotation

In most cases, the observers are rest on the rotating Earth. Since the Earth is almost considered as a rigid body, the position vector of such observer is expressed[1] as

$$\vec{r}_0 = \mathcal{R}^{-1}\tilde{\vec{r}} \tag{7.1}$$

where $\vec{r}_0$ is the position vector in a certain inertial coordinate system at an epoch t_0 and $\tilde{\vec{r}}$ is its expression referred to the rotating Earth at time t. As for the inertial coordinate system, we may adopt any kind of it. This is because that the Galilei's principle of relativity assures that the physical phenomenon are independent with the choice of inertial coordinate systems. However, from the viewpoint to avoid the unnecessary complexity, we usually adopt the equatorial coordinate system as such.

Anyway the position on the rotating Earth is described as

$$\tilde{\vec{r}} = \begin{pmatrix} \rho_N \cos\varphi \cos\lambda \\ \rho_N \cos\varphi \sin\lambda \\ \rho_Z \sin\varphi \end{pmatrix} \tag{7.2}$$

Here

$$\rho_N = N(\varphi) + h, \qquad \rho_Z = \left(1 - e^2\right) N(\varphi) + h, \tag{7.3}$$

where

$$N(\varphi) = \frac{a}{\sqrt{1 - e^2 \sin^2\varphi}} \tag{7.4}$$

[1] We take a convention that the matrix $\mathcal{R}$ expresses the rotation *from* the inertial frame *to* the frame fixed to the rotating Earth. This is due to the historical convention to express the precession-nutation matrix in this sense.

and φ, λ, and h are the geographic latitude, the longitude, and the height from the reference Earth spheroid of the observer, respectively[2] . The trio, (φ, λ, h), is called as the geodetic coordinates as a whole. The pair (a, e) is the equatorial radius and the eccentricity of the reference spheroid of the Earth. To specify the sphroid, we usually select not the eccentricity defined as

$$e \equiv \frac{a^2 - b^2}{a^2}, \tag{7.5}$$

but the flatenning factor defined as

$$f = \frac{a - b}{a}, \tag{7.6}$$

where b is the polar radius of the spheroid.

The typical values of a and f are

$$a = 6378136\text{m}, \qquad f = 1/298.257 \tag{7.7}$$

The rotation matrix $\mathcal{R}$ is called as the Earth rotation matrix, which is decomposed as the products of four matrices;

$$\mathcal{R} = \mathcal{W}\mathcal{S}\mathcal{N}\mathcal{P} \tag{7.8}$$

where $\mathcal{P}$ is the precession matrix, $\mathcal{N}$ is the nutation matrix, $\mathcal{S}$ is the sidereal rotation matrix, and $\mathcal{W}$ is the polar motion (or wobble) matrix.

Apart from $\mathcal{S}$, the other three matrices are close to the unit matrix. Actually, the differences from the unit matrix are of the order of 50"/y for the precession, 10" for the nutation, and only 0".3 for the polar motion.

Remark that the first two rotations, the precession and nutation, are common to all the celestial objects and independent with the location of the observer on the Earth. Therefore, two intermediate expressions of the observer's position have been frequently used:

$$\vec{r}_M \equiv \mathcal{P}\vec{r}, \qquad \vec{r}_T \equiv \mathcal{N}\vec{r}_M = \mathcal{N}\mathcal{P}\vec{r} \tag{7.9}$$

Historically, the intermediate expression $\vec{r}_M$ has been denoted as the position referred to the *mean equator and equinox of date*[3] , and another intermediate expression $\vec{r}_T$ has been denoted as the position referred to the *true equator and equinox of date.* While the original expression, $\vec{r}$, is denoted as the position referred to the *mean equator and equinox of the epoch.* Here the epoch is usualy taken as J2000.0, namely the instant when JD=2451545.0.

[2] In some Germany and/or geodetic literatures, the symbols B and L are used in place of φ and λ, respectively.

[3] The word *of date* means *at time t*, which is a variable.

7.1.1 Precession

The precession is the long-period part of the motion of the spin axis of the Earth. Since its period is so long as around 26000 years, its time variation looks almost constant. Thus it is named after the annual rate of *precession* of the spin axis.

The precession is expressed as a function of time in the form of a 323-sequence of the rotation matrix as

$$\mathcal{P}(t) = \mathcal{R}_{323}(-\zeta_A, \theta_A, -z_A) = \begin{pmatrix} -\sin z_A \sin\zeta_A + \cos z_A \cos\theta_A \cos\zeta_A & -\sin z_A \cos\zeta_A - \cos z_A \cos\theta_A \sin\zeta_A & -\cos z_A \sin\theta_A \\ \cos z_A \sin\zeta_A + \sin z_A \cos\theta_A \cos\zeta_A & \cos z_A \cos\zeta_A - \sin z_A \cos\theta_A \sin\zeta_A & -\sin\theta_A \sin\zeta_A \\ \sin\theta_A \cos\zeta_A & -\sin z_A \sin\theta_A & \cos\theta_A \end{pmatrix} \quad (7.10)$$

where the three rotational angles (ζ_A, θ_A, z_A) are denoted as the accumulated[4] precession angles in the equatorial coordinates. Remark that the inverse matrix is given as

$$(\mathcal{P})^{-1} = \mathcal{P}^t = \mathcal{R}_{313}(z_A, -\theta_A, \zeta_A) \quad (7.11)$$

According to the IAU (1976) precession formula, these angles are expressed as cubic polynomials of time as

$$\zeta_A = 2306".2181 T_{2000} + 0".30188 T_{2000}^2 + 0".017998 T_{2000}^3, \quad (7.12)$$

$$\theta_A = 2004".3109 T_{2000} - 0".42665 T_{2000}^2 - 0".041833 T_{2000}^3, \quad (7.13)$$

$$z_A = 2306".2181 T_{2000} + 1".09468 T_{2000}^2 + 0".018203 T_{2000}^3 \quad (7.14)$$

where T_{2000} is the time from the epoch J2000.0 (=JD2451545.0) measured in the unit of julian century.

7.1.2 Two-Time Form of Precession

Remark that the expression given in the precious subsection provides the rotation matrix from the epoch J2000.0 to the time t.

[4] The subscript A stands for the *accumulated*, which means that the quantities are obtained by integrating the instantenous rates of precession.

If one needs the precessional rotation from an arbitrary epoch t' to t, the matrix is rigorously[5] given as a product of two precessional ones as

$$\mathcal{P}(t' \to t) = \mathcal{P}(t' \to t_0)\mathcal{P}(t_0 \to t) = \mathcal{P}^{-1}(t')\mathcal{P}(t) = \mathcal{P}^t(t')\mathcal{P}(t) \tag{7.16}$$

However, this requires the computation of two matrices to obtain one matrix. In principle, any rotation is written as one rotational matrix. Therefore, one may want a compacter expression. Such expression is given in terms of the so-called *two-time* forms of the precession angles as

$$\mathcal{P}(T \to T+t) = \mathcal{R}_{323}\left(-\zeta_A(T,t), \theta_A(T,t), -z_A(T,t)\right) \tag{7.17}$$

where

$$\begin{pmatrix} \zeta_A \\ \theta_A \\ z_A \end{pmatrix}(T,t) = \begin{pmatrix} \zeta_{A1}(T) \\ \theta_{A1}(T) \\ z_{A1}(T) \end{pmatrix} t + \begin{pmatrix} \zeta_{A2}(T) \\ \theta_{A2}(T) \\ z_{A2}(T) \end{pmatrix} t^2 + \begin{pmatrix} \zeta_{A3} \\ \theta_{A3} \\ z_{A3} \end{pmatrix} t^3 \tag{7.18}$$

where

$$\zeta_{A1}(T) = z_{A1}(T) = 2306".2181 + 1".39656T - 0".000139T^2, \tag{7.19}$$

$$\zeta_{A2}(T) = 0".30188 - 0".000344T, \qquad \zeta_{A3} = 0".017998, \tag{7.20}$$

$$\theta_{A1}(T) = 2004".3109 - 0".85330T - 0".000217T^2, \tag{7.21}$$

$$\theta_{A2}(T) = -0".42665 - 0".000217T, \qquad \theta_{A3} = -0".041833, \tag{7.22}$$

$$z_{A2}(T) = 1".09468 + 0".000066T, \qquad z_{A3} = 0".018203. \tag{7.23}$$

where T and t denote the starting and ending times of the precessional rotation, and they are measured from J2000.0 in the unit of Julian century. Remark that

$$\begin{pmatrix} \zeta_A \\ \theta_A \\ z_A \end{pmatrix}(0, T_{2000}) = \begin{pmatrix} \zeta_A \\ \theta_A \\ z_A \end{pmatrix} \tag{7.24}$$

[5] We prefer this formulation since it is more rigorous than using the two-time formulas of the precession. The reason is that ithis rigorous form keeps the law of transition of precession matrix

$$\mathcal{P}(t_0 \to t_2) = \mathcal{P}(t_1 \to t_2)\mathcal{P}(t_0 \to t_1) \tag{7.15}$$

exactly.

7.1.3 Approximate Form of Precession

Now, consider the approximate form of the precession.

Assume that the precession angles are small, or that the time considered is sufficiently close to the epoch. Then, the precession matrix is expanded as

$$\mathcal{P} = \mathcal{I} + \Delta_1\mathcal{P} + \Delta_2\mathcal{P} + \cdots \tag{7.25}$$

Here the first order correction is given as

$$\Delta_1\mathcal{P} = \begin{pmatrix} 0 & -\phi_A & -\theta_A \\ \phi_A & 0 & 0 \\ \theta_A & 0 & 0 \end{pmatrix} \tag{7.26}$$

where

$$\phi_A \equiv \zeta_A + z_A \tag{7.27}$$

The second order correction is

$$\Delta_2\mathcal{P} = \frac{-1}{2}\begin{pmatrix} \phi_A^2 + \theta_A^2 & 0 & 0 \\ 0 & \phi_A^2 & 2z_A\theta_A \\ 0 & 2\zeta_A\theta_A & \theta_A^2 \end{pmatrix} \tag{7.28}$$

If we ignore the second and higer order terms, we obtain an approximate relation of angular components of the direction vector

$$\begin{pmatrix} -(\Delta_P\delta)\sin\delta\cos\alpha - (\Delta_P\alpha)\cos\delta\sin\alpha \\ -(\Delta_P\delta)\sin\delta\sin\alpha + (\Delta_P\alpha)\cos\delta\cos\alpha \\ (\Delta_P\delta)\cos\delta \end{pmatrix} \approx \begin{pmatrix} -\phi_A\cos\delta\sin\alpha - \theta_A\sin\delta \\ \phi_A\cos\delta\cos\alpha \\ \theta_A\cos\delta\cos\alpha \end{pmatrix} \tag{7.29}$$

which is transformed as

$$\Delta_P\alpha \approx \phi_A + \theta_A\tan\delta\sin\alpha, \qquad \Delta_P\delta \approx \theta_A\cos\alpha \tag{7.30}$$

This is the first approximate formula of precession. In this sense, ϕ_A and θ_A are sometimes called as the accumulated precessions in right ascension and in declination, respectively[6] .

Further, keep only the first terms in the expressions of ϕ_A and θ_A. Then, these are approximated as

$$\phi_A \approx 4612''.4362T_{2000} \equiv m_P T_{2000}, \qquad \theta_A \approx 2004''.3109T_{2000} \equiv n_P T_{2000}, \tag{7.31}$$

where m_P and n_P are called as the general precession in right ascension and in declination, respectively. Therefore,

$$\Delta_P\alpha \approx (m_P + n_P\tan\delta\sin\alpha)\,T_{2000}, \qquad \Delta_P\delta \approx (n_P\cos\alpha)\,T_{2000} \tag{7.32}$$

[6] Sometimes, the symbols M_P and N_P are used in place of ϕ_A and θ_A.

This is the second approximate form of the precession. This implies that the precession is approximately interpreted as a constant rotation around a fixed axis. The angular velocity vector of such constant rotation is obtained as

$$\vec{\omega}_P = \begin{pmatrix} 0 \\ n_P \\ -m_P \end{pmatrix} \tag{7.33}$$

whose magnitude is obtained as

$$\omega_P \equiv |\vec{\omega}_P| = \sqrt{n_P^2 + m_P^2} \approx 5029.0986\text{"jc}^{-1} \tag{7.34}$$

We call p the general precession in longitude. This is because the direction of the angular velocity vector is almost parallel to the ecliptic pole, and therefore, the precession is considered as a rotation along the longitude.

7.1.4 Precession in Ecliptic Coordinate System

Originally, the precessional quantities were described in terms of the ecliptic coordinate system. This was because the orbital motions of the Sun and the Moon, the major perturbers to the rotational motion of the Earth, are more compactly described in the ecliptic coordinate system. There the relation is given as

$$\vec{r}_{\text{ECL}} = \mathcal{P}_{\text{ECL}} \vec{r}_{\text{ECL0}} \tag{7.35}$$

where $\vec{r}_{\text{ECL}}$ is the position vector in the ecliptic coordinate system at time t, and $\vec{r}_{\text{ECL0}}$ is the position vector in the ecliptic coordinate system at an epoch t_0, The precession matrix in ecliptic coordinates, $\mathcal{P}_{\text{ECL}}$, is expressed by the 313-sequence as

$$\mathcal{P}_{\text{ECL}} = \mathcal{R}_{313}\left(\Pi_A, \pi_A, -\left(p_A + \Pi_A\right)\right) \tag{7.36}$$

The Euler angles, Π_A, π_A, and p_A, are called as the accumulated precession angles in the ecliptic coordinates. According to the IAU (1976) precession formula, these angles are expressed as as

$$\bar{p} \equiv \sin\pi_A \cos\Pi_A = -46\text{"}.8150 T_{2000} + 0\text{"}.05059 T_{2000}^2 + 0\text{"}.000344 T_{2000}^3, \tag{7.37}$$

$$\bar{q} \equiv \sin\pi_A \sin\Pi_A = 4''.1976 T_{2000} + 0\text{"}.19447 T_{2000}^2 - 0\text{"}.000179 T_{2000}^3, \tag{7.38}$$

$$p_A = 5029\text{"}.0966 T_{2000} + 1\text{"}.11113 T_{2000}^2 - 0\text{"}.000006 T_{2000}^3 \tag{7.39}$$

The first two are resolved approximately as

$$\Pi_A \approx 174°52'34''.982 - 869''.8089 T_{2000} + 0\text{"}.03536 T_{2000}^2, \tag{7.40}$$

$$\pi_A \approx 47".0029T_{2000} - 0".03302T_{2000}^2 + 0".000060T_{2000}^3, \tag{7.41}$$

However, these expressions are inaccurate such that we recommend the usage of $\bar{p}$ and $\bar{q}$ instead.

Anyway, we remark that

$$\pi_A \ll p_A \ll \Pi_A \tag{7.42}$$

Thus, we may apply the rewriting of the 313-sequence when the second Euler angle is small as

$$\mathcal{P}_{\mathrm{ECL}} = \mathcal{R}_3\left(-\left(p_A + \Pi_A\right)\right)\mathcal{R}_1\left(\pi_A\right)\mathcal{R}_3\left(\Pi_A\right) = \mathcal{R}_3\left(-p_A\right)\mathcal{Q}_{\mathrm{ECL}} \tag{7.43}$$

where

$$\mathcal{Q}_{\mathrm{ECL}} \equiv \mathcal{R}_{313}\left(\Pi_A, \pi_A, -\Pi_A\right)\begin{pmatrix} 1-\bar{q}^2\bar{r} & \bar{p}\bar{q}\bar{r} & -\bar{q} \\ \bar{p}\bar{q}\bar{r} & 1-\bar{p}^2\bar{r} & \bar{p} \\ \bar{q} & -\bar{p} & \bar{c} \end{pmatrix} \tag{7.44}$$

where

$$\bar{r} \equiv \frac{1}{1+\bar{c}}, \qquad \bar{c} \equiv \cos\pi_A = \sqrt{1-\left(\bar{p}^2+\bar{q}^2\right)} \tag{7.45}$$

Let us apply the transformation formula between the ecliptic and equatorial coordinate systems to both the position vectors at the epoch and at time t, respectively, as

$$\vec{r}_{\mathrm{ECL}} = \mathcal{R}_1\left(\varepsilon_A\right)\vec{r}, \qquad \vec{r}_{\mathrm{ECL0}} = \mathcal{R}_1\left(\varepsilon_0\right)\vec{r}_0, \tag{7.46}$$

where

$$\varepsilon_0 \equiv \left.\varepsilon_A\right|_{T_2000=0} \tag{7.47}$$

and

$$\varepsilon_A = 23°26'21".448 - 46".8150T_{2000} - 0".00059T_{2000}^2 + 0".001813T_{2000}^3, \tag{7.48}$$

Then

$$\mathcal{P} = \mathcal{R}_1\left(-\varepsilon_A\right)\mathcal{P}_{\mathrm{ECL}}\mathcal{R}_1\left(\varepsilon_0\right) \tag{7.49}$$

If we further rewrite it as

$$\mathcal{P} = \mathcal{R}_1\left(-\left(\varepsilon_A - \varepsilon_0\right)\right)\mathcal{R}_{131}\left(\varepsilon_0, p_A, -\varepsilon_0\right)\mathcal{Q} \tag{7.50}$$

where

$$\mathcal{Q} \equiv \mathcal{R}_1\left(-\varepsilon_A\right)\mathcal{Q}_{\mathrm{ECL}}\mathcal{R}_1\left(\varepsilon_A\right) \tag{7.51}$$

7.1.5 Nutation

The nutation is the short-period[7] part of the motion of the spin axis of the Earth.

The nutation is expressed as a function of time in the form of a 131-sequence of the rotation matrix as

$$\mathcal{N} = \mathcal{R}_{131}\left(\varepsilon_A, -\Delta\psi, -(\varepsilon_A + \Delta\varepsilon)\right) \tag{7.52}$$

where ε_A is one of the accumulated precession angles and is named as the obliquity of the ecliptic[8] . The two small angles $\Delta\psi$ and $\Delta\varepsilon$ are the nutation in the (ecliptic) longitude and in the obliquity, respectively.

According to the IAU (1976) precession formula, ε_A is expressed as a cubic polynomial of time as

$$\varepsilon_A = 23°26'21".448 - 46".8150 T_{2000} - 0".00059 T_{2000}^2 + 0".001813 T_{2000}^3, \tag{7.53}$$

where T_{2000} is the time from the epoch J2000.0 (=JD2451545.0) measured in the unit of julian century.

The nutation angles $(\Delta\psi, \Delta\varepsilon)$ are usually expressed as Fourier series as

$$\Delta\psi = \sum_j S_j \sin A_j, \qquad \Delta\varepsilon = \sum_j C_j \cos A_j, \tag{7.54}$$

where the arguments A_j are a linear combination of some angle variables, whic are called as the fundamental arguments.

Usually, the fundamental arguments are five angles, $(\ell, \ell', F, D, \Omega)$, which appeared in describing the main part of the orbital motion of the Moon. The main part is actually the three body problem of the Earth, the Moon, and the Sun. The angles are

- ℓ: the mean anomaly of the Moon,
- ℓ': the mean anomaly of the Sun[9] ,
- F: the mean arugument of latitude of the Moon,
- D: the mean elongation of the Moon from the Sun, namely the difference in the mean longitude, $D \equiv L - L'$, where L and L' are the mean longitude of he Moon and the Sun, respectively,

[7] The periods are equal to or less than around 18.6 years, which is the period of the motion of the node of Moon.

[8] The word *ecliptic* denotes the mean orbital plane of the Earth-Moon barycenter around the Sun. The obliquity of ecliptic means the secular component of the inclination of the Earth's orbit referred to the equatorial plane, which is defined as the plane orthogonal to the spin axis of the Earth.

[9] Regarded as a body moving around the Earth

- Ω: the mean longitude of the ascending node of the Moon on the ecliptic.

Remark that the coordinate origin is taken as the Earth so that we regard the Sun and the Moon move around the Earth.

According to the IAU (1980) nutation theory, these arguments are given as low order polynomials of time:

$$\ell = 134°57'46".733 + 1325^{\rm rev}198°52'02".633T_{2000} + 31".310T_{2000}^2 + 0".064T_{2000}^3 \quad (7.55)$$

$$\ell' = 357°31'39".804 + 99^{\rm rev}359°03'01".224T_{2000} - 0".577T_{2000}^2 - 0".012T_{2000}^3 \quad (7.56)$$

$$F = 93°16'18".877 + 1342^{\rm rev}82°01'03".137T_{2000} - 13".257T_{2000}^2 + 0".011T_{2000}^3 \quad (7.57)$$

$$D = 297°51'01".307 + 1236^{\rm rev}307°06'41".328T_{2000} - 6".891T_{2000}^2 + 0".019T_{2000}^3 \quad (7.58)$$

$$\Omega = 135°02'40".280 - 5^{\rm rev}134°08'10".539T_{2000} + 7".455T_{2000}^2 + 0".008T_{2000}^3 \quad (7.59)$$

where T_{2000} is the time measured from the epoch J2000.0 in the unit of julian century.

In the IAU (1980) nutation theory, the nutations are given as 106 Fourier terms with the arguments of the form of

$$A_j = n_j^{(\ell)}\ell + n_j^{(\ell')}\ell' + n_j^{(F)}F + n_j^{(D)}D + n_j^{(\Omega)}\Omega \quad (7.60)$$

where n_js are small integers in the range $|n_j| \leq 4$.

7.1.6 Approximate Form of Nutation

The principal nutations, which are the terms with the amplitude greater than 0".1, are

$$\begin{pmatrix} \Delta\psi \\ \Delta\varepsilon \end{pmatrix} = \begin{pmatrix} -17".2 \sin\Omega \\ +9".2 \cos\Omega \end{pmatrix} + \begin{pmatrix} -1".3 \sin 2L' \\ +0".6 \cos 2L' \end{pmatrix} + \begin{pmatrix} +0".2 \sin 2\Omega \\ -0".1 \cos 2\Omega \end{pmatrix}$$
$$+ \begin{pmatrix} -0".2 \sin 2L \\ +0".1 \cos 2L \end{pmatrix} + \begin{pmatrix} +0".1 \sin \ell' \\ 0 \end{pmatrix} + \begin{pmatrix} +0".1 \sin \ell \\ 0 \end{pmatrix} + \cdots \quad (7.61)$$

where $L' \equiv F - D + \Omega$ is the mean longitude of the Sun (around the Earth) and $L \equiv F + \Omega$ is the mean longitude of the Moon.

The nutation matrix is expanded as

$$\mathcal{N} = \mathcal{I} + \Delta_1\mathcal{N} + \Delta_2\mathcal{N} + \cdots \quad (7.62)$$

Here the first order correction is given as

$$\Delta_1\mathcal{N} = \begin{pmatrix} 0 & -\Delta\mu & -\Delta\nu \\ \Delta\mu & 0 & -\Delta\varepsilon \\ \Delta\nu & \Delta\varepsilon & 0 \end{pmatrix} \tag{7.63}$$

where

$$\Delta\mu \equiv \Delta\psi \cos\varepsilon_A, \qquad \Delta\nu \equiv \Delta\psi \sin\varepsilon_A \tag{7.64}$$

Note that the cosine and the sine of ε_A is almost constant as

$$\cos\varepsilon_A \approx 0.91748, \qquad \sin\varepsilon_A \approx 0.39778 \tag{7.65}$$

The second order correction is

$$\Delta_2\mathcal{N} = \frac{-1}{2}\begin{pmatrix} (\Delta\psi)^2 & 0 & 0 \\ 2\Delta\varepsilon\Delta\nu & [(\Delta\varepsilon)^2 - (\Delta\mu)^2] & \Delta\mu\Delta\nu \\ -2\Delta\varepsilon\Delta\mu & \Delta\mu\Delta\nu & [(\Delta\varepsilon)^2 + (\Delta\nu)^2] \end{pmatrix} \tag{7.66}$$

Remark that the order of magnitude is 10" for $\Delta_1\mathcal{N}$ and $(10")^2 \sim 1$ mas for $\Delta_2\mathcal{N}$.

Let us ignore the second and higer order terms, which are of the order of 1 mas. This approximation is sufficient for most purposes. Then, we obtain an approximate relation of angular components of the direction vector

$$\begin{pmatrix} -(\Delta_N\delta)\sin\delta\cos\alpha - (\Delta_N\alpha)\cos\delta\sin\alpha \\ -(\Delta_N\delta)\sin\delta\sin\alpha + (\Delta_N\alpha)\cos\delta\cos\alpha \\ (\Delta_N\delta)\cos\delta \end{pmatrix} \approx \begin{pmatrix} -\Delta\mu\cos\delta\sin\alpha - \Delta\nu\sin\delta \\ \Delta\mu\cos\delta\cos\alpha - \Delta\varepsilon\sin\delta \\ \Delta\nu\cos\delta\cos\alpha + \Delta\varepsilon\sin\delta \end{pmatrix} \tag{7.67}$$

which is transformed as

$$\Delta_N\alpha \approx \Delta\mu + \Delta\nu\sin\alpha\tan\delta - \Delta\varepsilon\cos\alpha\tan\delta, \qquad \Delta_N\delta \approx \Delta\nu\cos\alpha + \Delta\varepsilon\sin\alpha \tag{7.68}$$

This is the approximate formula of nutation.

7.1.7 Sidereal Rotation

The main part of the Earth rotation is the almost constant rotation around the spin axis of the Earth, which is called the sidereal[10] rotation. The sidereal rotation is expressed as

$$\mathcal{S} = \mathcal{R}_3(\Theta) \tag{7.69}$$

where Θ is the angle of sidereal rotation, which is historically called as the *Apparent Sidereal Time*, AST. Although AST and other *sidereal time*s are actually rotational angles, they

[10] Te word *sidereal* means *of stars*.

have been called as *time*. Remark that time units (hour, minute, second) are used in expressing the sidereal times.

The AST consists of two parts; the secular and the periodic. The secular part is called as the *Mean Sidereal Time*, MST, while the periodic part is called as the *Equation*[11] *of Equinoxes*, EE. Namely

$$\mathrm{AST} = \mathrm{MST} + \mathrm{EE} \tag{7.70}$$

Here the equation of equinoxes are mainly due to the nutation, which is given as

$$\mathrm{EE} = \Delta\psi \cos\varepsilon_A + 2.64\mathrm{mas}\sin\Omega + 0.63\mathrm{mas}\sin 2\Omega \tag{7.71}$$

Ocassionally, an adjective is attached to specify the meridian to which the sidereal time is referred. If it is referred to a local meridian, AST or MST are called as Local AST (LAST) or Local MST (LMST). If it is referred to the zero meridian, namely that of Greenwich, AST or MST are called as Greenwich AST (GAST) or Greenwich MST (GMST). The relations between GMST (or GAST) and LMST (or GAST) are

$$\mathrm{LAST} - \mathrm{GAST} = \mathrm{LMST} - \mathrm{GMST} = \lambda \tag{7.72}$$

where λ is the longitude of the meridian to which LAST (or LMST) is referred. Therefore

$$\mathrm{LAST} = \mathrm{LMST} + \mathrm{EE}, \qquad \mathrm{GAST} = \mathrm{GMST} + \mathrm{EE} \tag{7.73}$$

The expression[12] of GMST is given in the unit of second of time[13] as

$$\mathrm{GMST} = 24110.54841 + 8640184.812866 T_U + 0.093104 T_U^2 - 6.2 \times 10^{-6} T_U^3 \tag{7.74}$$

where T_U is the universal time[14] measured in UT Julian century since J2000.0 UT. Preceisely speaking, the universal time used here is the first variation[15] of the universal time, UT1. To make this clear, the symbol GMST1 is sometimes used instead of GMST.

---

[11] The word *equation* means the difference in this context

[12] You may feel it somewhat peculiar. This is due to a historical reason that the sidereal time as well as the universal time had been regarded not as a rotational angle but as a measure of time. The naming *sidereal time* or *universal time* itself is a good illustration supporting this fact.

[13] Namely the second means $2\pi/(360 \times 3600) \approx 4.848 \times 10^{-6}$ radian in this context.

[14] The universal time is another measure of rotation angle of the Earth, whose unit is close to the so-called mean solar second.

[15] There exists the zero-th variation, UT0, which is the direct observable of the rotation angle of the Earth. UT1 is obtained from UT0 by removing the effect of polar motion.

7.1.8 Polar Motion

In general, the rotational z-axis[16] , *pole* if roughly speaking, of a rotating body is not fixed to the figure axis[17] of the body. The difference is called as the *polar motion* or the *wobble*. The polar motion is expressed in two components x_P and y_P. The rotation associated to the polar motion is expressed as

$$\mathcal{W} = \mathcal{R}_2\left(-x_P\right)\mathcal{R}_1\left(-y_P\right) \tag{7.75}$$

which is sometimes called as the wobble matrix. The magnitude of the polar motion is around 0".3. The polar motion mainly consists of two components; the annual and the Chandler parts. The period of the latter component is roughly 14 months. Since the polar motion is so small, we may ignore the second order terms and simplify the relation of angular components of the direction vector as

$$\Delta_W\varphi \approx x_P\cos\lambda - y_P\sin\lambda, \qquad \Delta_W\lambda \approx \left(x_P\sin\lambda + y_P\cos\lambda\right)\tan\varphi \tag{7.76}$$

This is the approximate formula of polar motion.

7.2 Actual Reference Frames

7.2.1 Celestial Reference Frame

Currently the reference frame covering the whole universe, *celestial reference frame*, is constructed by a table of direction vectors of quasars. Since quasars are thought to be very distant from us, we may ignore their proper motions and their parallactic effects. In other words, their apparent direction vectors, if we remove corrections due to the observer effects such as the aberration, can be thought as fixed directions in the universe.

An example of such table of direction vectors of quasars is the International Celestial Reference Frame (ICRF). It is defined and maintained by the International Earth Rotation Service (IERS). The IERS is an international academic service under the Federation of Astronomical and Geophysical Data Analysis Services (FAGS), which is a joint service of two international scientific unions, the International Astronomical Union (IAU) and the International Union of Geodesy and Geophysics (IUGG).

7.2.2 Galactic Reference Frame

The concept of *Galactic reference frame* has been vague. The reason is that the center of our galaxy, which should be the coordinate origin of the Galactic reference farme, is not so

[16] The z-axis of the reference frame attached to the body

[17] The axis of the maximum moment of inertia. It is sometimes called as the C-axis.

rigorously determined. The same is true for the so-called galactic plane, which should be the equatorial plane of the Galactic reference frame. Therefore, it is common to keep the coordinate origin as the barycenter of the solar system and to adopt a different orientation other than that of the solar-system-barycentric reference frame.

The usual expression of position in the galactic reference frame is polar coordinates (r, b, ℓ):

$$\vec{r}_P^{(\mathrm{GAL})} = r \begin{pmatrix} \cos b \cos \ell \\ \cos b \sin \ell \\ \sin b \end{pmatrix} \tag{7.77}$$

Here r is the distance from the solar-system-barycenter, b is the galactic latitude, and ℓ is the galactic longitude.

7.2.3 Solar-System-Barycentric Reference Frame

Another important reference frame is the solar-system-barycentric (SSB) reference frame. Its coordinate origin is the barycenter of the solar system defined as

$$\sum_J M_J \vec{r}_J^{(\mathrm{SSB})} = 0 \tag{7.78}$$

where the suffix J runs all the solar system bodies; the Sun, nine major planets, a lot of minor planets (or asteroids), the Moon and a number of natural satellites, comets, rings, and other minute bodies in the solar system.

There are four major options in expressing the coordinates in this reference frame

1. the equatorial rectangular coordinates, (X, Y, Z),
2. the equatorial polar coordinates, (r, α, δ),
3. the ecliptic rectangular coordinates, (U, V, W), and
4. the ecliptic polar coordinates, (r, λ, β).

Here r is called as the radius vector, α is the right ascension, δ is the declination, β is the celestial latitude, and λ is the celestial longitude. These three are connected by the relation

$$\vec{r}^{(\mathrm{SSB-EQ})} = \mathcal{R}_1 \left(\varepsilon_A\right) \vec{r}^{(\mathrm{SSB-ECL})} \tag{7.79}$$

Here

$$\vec{r}^{(\mathrm{SSB-EQ})} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = r \begin{pmatrix} \cos \delta \cos \alpha \\ \cos \delta \sin \alpha \\ \sin \delta \end{pmatrix} \tag{7.80}$$

is the position vector in the equatorial reference frame. and

$$\vec{r}^{(\mathrm{SSB-ECL})} = \begin{pmatrix} U \\ V \\ W \end{pmatrix} = r \begin{pmatrix} \cos\beta\cos\lambda \\ \cos\beta\sin\lambda \\ \sin\beta \end{pmatrix} \tag{7.81}$$

is the position vector in the ecliptic reference frame, and ε_A is the accumulated obliquity of the ecliptic .

7.2.4 Geocentric Reference Frame

The geocentric reference frame is the reference frame whose coordinate origin is the geocenter, the barycenter of the Earth;

$$\vec{r}_P^{(\mathrm{GEO-XXX})} = \vec{r}_P^{(\mathrm{SSB-XXX})} - \vec{r}_{\mathrm{Earth}}^{(\mathrm{SSB-XXX})} \tag{7.82}$$

Here $\vec{r}_{\mathrm{Earth}}$ is the position vector of the geocenter in the same solar-system-barycentric coordinate system (SSB-XXX) such as $\vec{r}^{(\mathrm{SSB-EQ})}$

Just as the same as in the solar-system-barycentric case, there are two major options for the orientation of reference frame; the equatorial- and the ecliptic ones. These three are connected by the relation

$$\vec{r}^{(\mathrm{GEO-EQ})} = \mathcal{R}_1(\varepsilon_A)\,\vec{r}^{(\mathrm{GEO-ECL})} \tag{7.83}$$

Here

$$\vec{r}^{(\mathrm{GEO-EQ})} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = r \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix} \tag{7.84}$$

is the position vector in the geocentric equatorial reference frame . and

$$\vec{r}^{(\mathrm{GEO-ECL})} = \begin{pmatrix} U \\ V \\ W \end{pmatrix} = r \begin{pmatrix} \cos\beta\cos\lambda \\ \cos\beta\sin\lambda \\ \sin\beta \end{pmatrix} \tag{7.85}$$

is the position vector in the geocentric ecliptic reference frame. Again ε_A is the accumulated obliquity of the ecliptic .

7.2.5 Topocentric Reference Frame

The *topocenteric* reference frame is the reference frame whose coordinate origin is the observer him/herself. Usually the observer is rest on the Earth. Thus we will deal with those fixed to the rotating Earth.

There are two major options expressing the topocentric reference frame. The one is the local equatorial reference frame and the other is the horizontal reference frame.

The local equatorial reference frame is defined as the reference frame such that

$$\vec{r}_P^{(\mathrm{TOP})} = \rho \begin{pmatrix} \cos\delta\cos H \\ -\cos\delta\sin H \\ \sin\delta \end{pmatrix} = \vec{r}_P^{(\mathrm{OLD})} - \vec{r}_{\mathrm{OBS}}^{(\mathrm{OLD})} \tag{7.86}$$

where

$$H = \Theta - \alpha_{\mathrm{App}} \tag{7.87}$$

is the hour angle.

The horizontal reference frame[18] is defined as the reference frame such that

1. the coordinate origin is the location of the observer on the Earth,
2. the x-y plane is the horizontal plane, i.e. the plane vertical to the zenith, and
3. the x-axis is toward the north[19]

The position vector in the horizontal reference frame is expressed as a three dimensional polar coordinate system (ρ, a, A) where ρ is the (topocentric) distance, a is the altitude (angle), and A is the azimuth. Remark that the anlge A is measured *clockwise* from the north direction. Sometimes the zenith distance[20]

$$z \equiv \frac{\pi}{2} - a \tag{7.88}$$

is used instead of a. Also the pair of symbols (Alt, Az) or (Elv, Az) is used instead of (a, A).

The coordinate transformation is given as

$$\vec{r}_{\mathrm{TOP}} = \rho \begin{pmatrix} \cos a\cos A \\ -\cos a\sin A \\ \sin a \end{pmatrix} = \mathcal{R}_2\left(\frac{\pi}{2} - \varphi_{\mathrm{OBS}}\right)\left(\tilde{\vec{r}} - \tilde{\vec{r}}_{\mathrm{OBS}}\right) \tag{7.89}$$

[18] Sometimes the horizontal reference frame is called as the NEZ-reference frame since its three axes are the North, the East, and the Zenith.

[19] There is no necessity to take the north direction as x-axis. This is a historical convention caused by most of ancient observers lived in the Northern Hemisphere of the Earth.

[20] Actually z is an angle.

where $\varphi_{\rm OBS}$ is the geographic latitude of the observer and

$$\tilde{\vec{r}}_{\rm OBS} = \begin{pmatrix} \rho_N \cos\varphi_{\rm OBS} \cos\lambda_{\rm OBS} \\ \rho_N \cos\varphi_{\rm OBS} \sin\lambda_{\rm OBS} \\ \rho_Z \sin\varphi_{\rm OBS} \end{pmatrix} \tag{7.90}$$

is the position vector of the observer in the terrestrial reference frame.

7.2.6 Satellitocentric Reference Frame

7.3 Astronomical Ephemeris

7.4 Refraction

7.5 Least Square Method

Chapter 8

APPENDICES

8.1 Inverse Problem on Spheroidal Coordinates

8.1.1 Equation of Spheroidal Latitude

Consider how to obtain the spheroidal coordinates, (φ, λ, h), when the rectangular coordinates, (x, y, z), are given. Here we list again the relation between these;

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho_N \cos\varphi \cos\lambda \\ \rho_N \cos\varphi \sin\lambda \\ \rho_Z \sin\varphi \end{pmatrix} \tag{8.1}$$

where

$$\rho_N = N(\varphi) + h, \qquad \rho_Z = \left(1 - e^2\right) N(\varphi) + h, \tag{8.2}$$

and

$$N(\varphi) \equiv \frac{a}{d(e, \varphi)}, \qquad d(e, \varphi) \equiv \sqrt{1 - e^2 \sin^2 \varphi} \tag{8.3}$$

First, we note that λ is easily obtained as

$$\lambda = \mathrm{atan2}(y, x) \tag{8.4}$$

The pair (φ, h) is obtained by solving a following nonlinear equation with respect to φ first;

$$R \sin\varphi - Z \cos\varphi = \frac{e^2 \sin\varphi \cos\varphi}{d(e, \varphi)} \tag{8.5}$$

Here

$$R \equiv \frac{\sqrt{x^2 + y^2}}{a} \geq 0, \qquad Z \equiv \frac{z}{a} \tag{8.6}$$

are parameters. The equation was obtained from

$$\sqrt{x^2+y^2} = \{N(\varphi)+h\}\cos\varphi, \qquad z = \left\{\left(1-e^2\right)N(\varphi)+h\right\}\sin\varphi \tag{8.7}$$

by cancelling h and the nondimensionization, namely the division by a.

Remark that the above nonlinear equation reduces to a quartic equation[1] . Introduce a new variable

$$\tau \equiv \tan\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{e'\tan\varphi}\right)\right] \tag{8.8}$$

where its range is

$$-1 \le \tau \le 1 \tag{8.9}$$

Then

$$\cos\varphi = \frac{2e'\tau}{d'}, \qquad \sin\varphi = \frac{1-\tau^2}{d'}, \qquad d(e,\varphi) = \sqrt{1-e^2\sin^2\varphi} = \frac{e'\left(1+\tau^2\right)}{d'}, \tag{8.10}$$

where

$$d' = \sqrt{\left(1+\tau^2\right)^2 - 4e^2\tau^2} \tag{8.11}$$

By substituting these into Eq.(8.5), we obtain

$$R\left(\frac{1-\tau^2}{d'}\right) - Z\left(\frac{2e'\tau}{d'}\right) = e^2\left(\frac{1-\tau^2}{d'}\right)\left(\frac{2e'\tau}{d'}\right)\left(\frac{d'}{e'\left(1+\tau^2\right)}\right) \tag{8.12}$$

This is rewritten as

$$R\left(1-\tau^2\right) - 2e'Z\tau - 2e^2d'\tau\left(\frac{1-\tau^2}{1+\tau^2}\right) = 0 \tag{8.13}$$

which finally leads to a quartic equation with respect to τ as

$$R\tau^4 + U\tau^3 + V\tau - R = 0 \tag{8.14}$$

where

$$U = 2\left(e'Z - e^2\right), \qquad V = 2\left(e'Z + e^2\right) \tag{8.15}$$

This is the equation of spheroidal latitude. It is one of the fundamental equations of geodesy.

[1] A careless transformation such as $t = \tan(\varphi/2)$ leads to an eighth order equation.

It is well known that a quartic equation is rigorously solved by Ferrari's formula. However, we need only one real root. The acutual procedure[2] begins by preparing the following quantities;

$$S = e'Z, \qquad T = e^2, \qquad U = 2(S-T), \qquad V = 2(S+T),$$

$$P = \frac{1}{3}(4R^2 + UV), \qquad Q = 8STR, \qquad D = P^3 + Q^2,$$

$$J = \left(\sqrt[3]{Q+\sqrt{D}}\right)^2, \qquad K = \frac{2QJ}{P^2+PJ+J^2}, \qquad L = \sqrt{\frac{U^2}{4} + KR},$$

$$G = \frac{U}{2} + L, \qquad C = \frac{R^2V + KG}{L}, \qquad W = \frac{C}{G + \sqrt{G^2 + C}}. \tag{8.16}$$

Here

$$W \equiv R\tau \tag{8.17}$$

Then the solution is given as

$$\varphi = \text{atan2}\left((R^2 - W^2), 2e'RW\right), \tag{8.18}$$

8.1.2 Newton Method

However, we prefer using an iterative procedure to solve the equation of spheroidal latitude since it is sure and faster. We list the equation once again:

$$f_G(\tau) \equiv R\tau^4 + U\tau^3 + V\tau - R = 0 \tag{8.19}$$

where

$$S = e'Z, \qquad T = e^2, \qquad U = 2(S-T), \qquad V = 2(S+T) \tag{8.20}$$

Thanks to the symmetry of the original equation, Eq.(8.5), we may assume that $Z \geq 0$, and therefore, $S \geq 0$ without loss of generality. In that case,

$$0 \leq R, \qquad 0 \leq S, \qquad 0 < T, \qquad 0 < V, \tag{8.21}$$

while U can be positive or negative. Since

$$f_G(0) = -R \leq 0, \qquad f_G(1) = U + V = 4S \geq 0, \tag{8.22}$$

[2] The procedure shown here is a rewriting of the exact solution given in Borkowski(1989), which covers not all but a major portion of possible cases. The rewriting avoids cancellations of similar quantities.

we know that there exists at least one real solution in the range $0 \le \tau \le 1$. The first derivative of f_G is

$$f'_G(\tau) = 4R\tau^3 + 3U\tau^2 + V \tag{8.23}$$

Thus, the Newton corrector becomes

$$\tau^*(\tau) \equiv \tau - \frac{f_G(\tau)}{f'_G(\tau)} = \frac{R\,(3\tau^4+1) + 2U\tau^3}{4R\tau^3 + 3U\tau^2 + V} \tag{8.24}$$

The second derivative of f_G is

$$f''_G(\tau) = 12R\tau^2 + 6U\tau = 6\tau(2R\tau + U) \tag{8.25}$$

Thus the zeros of f''_G are $\tau = 0$ and

$$\tau_D = \frac{-U}{2R}. \tag{8.26}$$

In considering the starter, we separate the problem into three cases depending on the value of U:

1. Case A: $U \ge 0$

   In this case, the signature of f''_G is nonnegative in the interval considered as

   $$f''_G(\tau) \ge 0, \qquad \text{for} \qquad 0 \le \tau \le 1 \tag{8.27}$$

   Thus, the Newton method is stable if starting from an upper bound. Such an example is given by a minimization form

   $$\tau_0 = \min\left(\tau^*(0), \tau^*\left(\frac{1}{2}\right), \tau^*(1)\right) \tag{8.28}$$

   where

   $$\tau^*(0) = \frac{R}{V}\,, \qquad \tau^*\left(\frac{1}{2}\right) = \frac{19R + 4U}{8R + 12U + 16V}\,, \qquad \tau^*(1) = \frac{4R + 2U}{4R + 3U + V} \tag{8.29}$$

2. Case B: $U \le -2R$

   In this case, the signature of f''_G is nonpositive in the interval considered as

   $$f''_G(\tau) \le 0, \qquad \text{for} \qquad 0 \le \tau \le 1 \tag{8.30}$$

   Thus, the Newton method is stable if starting from a lower bound, whose example is given by a maximization form

   $$\tau_0 = \max\left(\tau^*(0), \tau^*\left(\frac{1}{2}\right), \tau^*(1)\right) \tag{8.31}$$

3. Case C: $-2R < U < 0$

   In this case, the signature of f''_G changes in the interval $[0, 1]$. In order to see the signature near the solution, we compute the value of f_G at τ_D as

$$D = f_G(\tau_D) \tag{8.32}$$

   and we separate the case into two subcases depending on the signature of D:

   (a) Subcase C1: $D \geq 0$

   In this subcase, the solution is in the subinterval $[0, \tau_D]$. In this subinterval, f''_G is nonpositive. Thus the Newton method is stable if starting from a lower bound. Such an example is given by a maximization form

$$\tau_0 = \max(\tau^*(0), \tau^*(\tau_D)) \tag{8.33}$$

   (b) Subcase C2: $D < 0$

   In this subcase, the solution is in the subinterval $(\tau_D, 1]$. In this subinterval, f''_G is nonnegative. Thus the Newton method is stable if starting from an upper bound. An example is the minimization form

$$\tau_0 = \min(\tau^*(\tau_D), \tau^*(1)) \tag{8.34}$$

Anyway, 3 or 4 iterations will be enough since the Newton method is of the second order. After the solution τ is obtained, φ and h are computed as

$$\varphi = \text{atan2}\left((1 - \tau^2), 2e'\tau\right), \tag{8.35}$$

$$h = \frac{a}{d'}\left[2Re'\tau + Z\left(1 - \tau^2\right) - e'\left(1 + \tau^2\right)\right] \tag{8.36}$$

where

$$d' = \sqrt{(1 + \tau^2)^2 - 4e^2\tau^2} \tag{8.37}$$

Refer the example shown in Table8.1

8.2 Other Sets of Keplerian Elements

The standard set of Keplerian elements $(a, e, I, \Omega, \omega, T)$ is inefficient in some limiting cases; the case I is small, the case e is small, and the case both e and I are small.

Table 8.1: Example Solution of Spheroidal Latitude Equation

Formula	Numerical Example
Parameters	
a	6378136m
$1/f$	298.257
Preparation I	
f	0.003352813177897
$e' = 1 - f$	0.996647186822103
$e^2 = 2f - f^2$	0.006694384999588
Rectangular Coordinates	
x	3912960.228939990
y	2259148.641506802
z	4488054.795103548
Preparation II	
$R = \sqrt{x^2 + y^2}/a$	0.708404035758034
$U = 2(e'Z - e^2)$	1.389217629689824
$V = 2(e'Z + e^2)$	1.415995169688176
Newton Corrector	
$\tau^*(\tau) \equiv (3R\tau^4 + 2U\tau^3 + R) \,/\, (4R\tau^3 + 3U\tau^2 + V)$	
Starter	
$\tau^*(0) = R/V$	0.666731046899295
$\tau^*(1/2) = (19R + 4U)/(8R + 12U + 16V)$	0.422648483405402
$\tau^*(1) = (4R + 2U)/(4R + 3U + V)$	0.500287042585064
Newton Iteration	
$\tau_0 = \min(\tau^*(0), \tau^*(1/2), \tau^*(1))$	$\underline{0.42}2648483405402$
$\tau_1 = \tau^*(\tau_0)$	$\underline{0.415}256056412539$
$\tau_2 = \tau^*(\tau_1)$	$\underline{0.41519757}2767164$
$\tau_3 = \tau^*(\tau_2)$	$\underline{0.415197569162213}$
Translation	
$d' = \sqrt{(1 + \tau^2)^2 - 4e^2\tau^2}$	1.170418670250952
$h = a\left[2Re'\tau + (1 - \tau^2)\, Z - e'\,(1 + \tau^2)\right]/d'$	$\underline{999.999999999}4216$ m
$\varphi = \tan^{-1}\left[(1 - \tau^2)\,/(2e'\tau)\right]$	$\underline{44^\circ.9999999999999}9$

Note: The correct digits are underlined. In selecting the starter, the case A, $U \geq 0$, is applied. The error in h is caused by the subtraction of similar quantities of the order of $a \sim 6 \times 10^7$ m, which is unavoidable. In other words, the relative error $|\delta h/(a + |h|)|$ is satisfactory small as of the order of 10^{-15}.

8.2.1 Low Inclination Orbit

When I is small, the procedures computing the position and velocity in the orbital plane remain well defined. However, the orientation of the orbital plane is ill-determined.

In this case, the trio (Ω, I, ω) is usually replaced by a new subset of variables (p, q, ϖ) where

$$p \equiv \tan\frac{I}{2}\cos\Omega, \qquad q \equiv \tan\frac{I}{2}\sin\Omega, \qquad \varpi \equiv \Omega + \omega \tag{8.38}$$

Then the rotation matrix specifying the orbital plane is rewritten as

$$\mathcal{R} = \mathcal{Q}\mathcal{R}_3\left(-\varpi\right) \tag{8.39}$$

where

$$\mathcal{Q} = \mathcal{I} - \frac{2}{1+p^2+q^2}\begin{pmatrix} q^2 & pq & q \\ pq & p^2 & p \\ -q & -p & p^2+q^2 \end{pmatrix} \tag{8.40}$$

Another useful subset is $(\bar{p}, \bar{q}, \varpi)$ where

$$\bar{p} \equiv \sin I \cos\Omega, \qquad \bar{q} \equiv \sin I \sin\Omega, \tag{8.41}$$

In this case, the above $\mathcal{Q}$ becomes

$$\mathcal{Q} = \begin{pmatrix} 1-\bar{q}^2\bar{r} & -\bar{p}\bar{q}\bar{r} & -\bar{q} \\ -\bar{p}\bar{q}\bar{r} & 1-\bar{p}^2\bar{r} & -\bar{p} \\ \bar{q} & \bar{p} & \bar{c} \end{pmatrix} \tag{8.42}$$

where

$$\bar{r} \equiv \frac{1}{1+\bar{c}}, \qquad \bar{c} \equiv \sqrt{1-\left(\bar{p}^2+\bar{q}^2\right)} \tag{8.43}$$

8.2.2 Nearly Circular Orbit

We frequently encounter with the nearly circular orbits. Most of the artificial satellites around the Earth are good examples.

When e is small, the orientation of the orbital plane remains well-defined. However, the direction of pericenter becomes ill-determined. Thus the procedures computing the position and velocity in the orbital plane for the elliptic case needs a modification.

In this case, all the anomalies, (f, M, E), must be replaced by the corresponding arguments of latitude, (v, N, D), where

$$v \equiv f + \omega, \qquad N \equiv M + \omega, \qquad D \equiv E + \omega, \tag{8.44}$$

Also the position components on the orbital plane are changed from those referred to the pericenter, (ξ, η), to those referred to the node

$$\bar{\xi} \equiv \xi \cos\omega - \eta \sin\omega, \qquad \bar{\eta} \equiv \xi \sin\omega + \eta \sin\omega. \tag{8.45}$$

As for Keplerian elements, the trio (e, ω, T) or (e, ω, M_0) is usually replaced by a new subset of variables (h, k, N_0) where

$$h \equiv e \cos\omega, \qquad k \equiv e \sin\omega, \qquad N_0 \equiv M_0 + \omega \tag{8.46}$$

Then the Kepler's equation is rewritten in the extended form as

$$D - h \sin D + k \cos D = N \tag{8.47}$$

where the eccentric argument of latitude, D, is solved for when the mean argument of latitude

$$N = n\,(t - t_0) + N_0 \tag{8.48}$$

and the components of the eccentricity vector referred to the node, (h, k) are given. The solution of the extended Kepler's equation will be discussed later. Remark that it is essentially different from that of the standard one.

Once the Kepler's equation is solved, the position components referred to the node are obtaiend as

$$\bar{\xi} = a\left[\left(1 - jk^2\right)\cos D + jhk \sin D - h\right], \tag{8.49}$$

$$\bar{\eta} = a\left[jhk \cos D + \left(1 - jh^2\right)\sin D - k\right] \tag{8.50}$$

where

$$j \equiv \frac{1}{1 + e'}, \qquad e' = \sqrt{1 - (h^2 + k^2)} \tag{8.51}$$

The corresponding velocity components are expressed as

$$\frac{\mathrm{d}\bar{\xi}}{\mathrm{dt}} = a\left[-\left(1 - jk^2\right)\sin D + jhk \cos D\right]\frac{\mathrm{d}D}{\mathrm{dt}}, \tag{8.52}$$

$$\frac{\mathrm{d}\bar{\eta}}{\mathrm{dt}} = a\left[-jhk \sin D + \left(1 - jh^2\right)\cos D\right]\frac{\mathrm{d}D}{\mathrm{dt}} \tag{8.53}$$

where the time derivative of D is given from the extended Kepler's equation as

$$\frac{\mathrm{d}D}{\mathrm{dt}} = \frac{n}{1 - h \cos D - k \sin D} \tag{8.54}$$

The final forms of the position and velocity are gievn by

$$\begin{pmatrix} \vec{r} & \vec{v} \end{pmatrix} = \mathcal{R}_3\left(-\Omega\right)\mathcal{R}_1\left(-I\right)\begin{pmatrix} \bar{\xi} & \mathrm{d}\bar{\xi}/\mathrm{dt} \\ \bar{\eta} & \mathrm{d}\bar{\eta}/\mathrm{dt} \\ 0 & 0 \end{pmatrix} \tag{8.55}$$

8.2.3 Nealy Circular Low Inclination Orbit

In some cases, we must deal with the orbits whose eccentricity and inclination are both small. Typical cases are the orbits of major planets in the ecliptic coordinates. So are the most of natural satellites referred to the equatorial plane of the mother planet.

When both e and I are small, neither the procedures computing the position and velocity in the orbital plane nor the orientation of the orbital plane is well-determined. In this case, the directions of pericenter and node are both ill-defined. Thus all the angles must be referred from the fiducial points. Namely all the anomalies, (f, M, E), must be replaced by the corresponding longitude, (λ, L, Λ), where

$$\lambda \equiv f + \varpi, \qquad L \equiv M + \varpi, \qquad \Lambda \equiv E + \varpi, \tag{8.56}$$

where

$$\varpi \equiv \Omega + \omega \tag{8.57}$$

Also the position components on the orbital plane must be changed from those referred to the pericenter, (ξ, η), to those referred to the departure point[3]

$$\tilde{\xi} \equiv \xi \cos \varpi - \eta \sin \varpi, \qquad \tilde{\eta} \equiv \xi \sin \varpi + \eta \sin \varpi. \tag{8.58}$$

As for Keplerian elements, the quartet $(e, I, \Omega, \omega, T)$ is usually replaced by a new subset of variables $\left(\tilde{h}, \tilde{k}, p, q, L_0\right)$ where

$$\tilde{h} \equiv e \cos \varpi, \qquad \tilde{k} \equiv e \sin \varpi, \qquad L_0 \equiv M_0 + \varpi \tag{8.59}$$

and

$$p \equiv \tan \frac{I}{2} \cos \Omega, \qquad q \equiv \tan \frac{I}{2} \sin \Omega, \tag{8.60}$$

or

$$\bar{p} \equiv \sin I \cos \Omega, \qquad \bar{q} \equiv \sin I \sin \Omega, \tag{8.61}$$

Then the Kepler's equation is rewritten in the extended form as

$$\Lambda - \tilde{h} \sin \Lambda + \tilde{k} \cos \Lambda = L \tag{8.62}$$

where the eccentric longitude, Λ, is solved for when the mean longitude

$$L = n\,(t - t_0) + L_0 \tag{8.63}$$

[3] The *departure point* is the foot of x-axis onto the orbit.

and the components of the eccentricity vector referred to the departure point, $(\tilde{h}, \tilde{k})$ are given. The solution of the extended Kepler's equation will be discussed later.

Once the extended Kepler's equation is solved, the position components referred to the departure point are obtaiend as

$$\tilde{\xi} = a\left[\left(1 - j\tilde{k}^2\right)\cos\Lambda + j\tilde{h}\tilde{k}\sin\Lambda - \tilde{h}\right], \tag{8.64}$$

$$\tilde{\eta} = a\left[j\tilde{h}\tilde{k}\cos\Lambda + \left(1 - j\tilde{h}^2\right)\sin\Lambda - \tilde{k}\right] \tag{8.65}$$

where j is the same as given in the precious subsection. The corresponding velocity components are expressed as

$$\frac{\mathrm{d}\tilde{\xi}}{\mathrm{dt}} = a\left[-\left(1 - j\tilde{k}^2\right)\sin\Lambda + j\tilde{h}\tilde{k}\cos\Lambda\right]\frac{\mathrm{d}\Lambda}{\mathrm{dt}}, \tag{8.66}$$

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{dt}} = a\left[-j\tilde{h}\tilde{k}\sin\Lambda + \left(1 - j\tilde{h}^2\right)\cos\Lambda\right]\frac{\mathrm{d}\Lambda}{\mathrm{dt}} \tag{8.67}$$

where the time derivative of Λ is given from the extended Kepler's equation as

$$\frac{\mathrm{d}\Lambda}{\mathrm{dt}} = \frac{n}{1 - \tilde{h}\cos\Lambda - \tilde{k}\sin\Lambda} \tag{8.68}$$

The final forms of the position and velocity are gievn by

$$\begin{pmatrix} \vec{r} & \vec{v} \end{pmatrix} = \mathcal{Q}\begin{pmatrix} \tilde{\xi} & \mathrm{d}\tilde{\xi}/\mathrm{dt} \\ \tilde{\eta} & \mathrm{d}\tilde{\eta}/\mathrm{dt} \\ 0 & 0 \end{pmatrix} \tag{8.69}$$

where $\mathcal{Q}$ is the same as given in Eq.(8.40), which expresses the rotation from the x-axis to the departure point.

8.3 Solving Kepler's Equation

The Kepler's equation is the first nonlinear equation appeared in the history. Its closed form solution is not known. Thus, all the solutions are of numerical and therefore approximate nature. The standard scheme is the combination of some starting procedures and the correction formulas.

8.3.1 Elliptic Case

First, consider the elliptic case, $0 \le e < 1$. The Kepler's equation is

$$f(E) \equiv E - e \sin E - M = 0 \tag{8.70}$$

where E is the variable to solve for. By the symmetry and the periodicity of 2π of the equation[4] , we can assume that $0 \le M < \pi$ and therefore $0 \le E < \pi$ without loss of generality.

Remark that

$$f(0) = -M \le 0, \qquad f(\pi) = \pi - M > 0 \tag{8.73}$$

Since $f(E)$ is continuous, we know that there is at least one real solution in the interval $[0, \pi)$. Further, the first derivative is nonnegative as

$$f'(E) = 1 - e \cos E \ge 0 \tag{8.74}$$

since $e < 1$. Thus the function $f(E)$ is monotonically increasing. Therefore, it has only one solution in the interval $[0, \pi)$. This means that, if we find an approximate solution in the interval by a certain procedure, the approximate solution is sufficiently close to the true solution.

As for the correction formula, we adopt the Newton formula,

$$E \to E^*(E) \equiv E - \frac{f(E)}{f'(E)} = \frac{M + e(\sin E - E \cos E)}{1 - e \cos E} \tag{8.75}$$

Remark that it is of the second order. Namely the effective digits of the approximate solution will be roughly doubled by each iteration if the iteration process converges[5] . In other words, if you start from an approximate solution with the error 0.1 or so, then the number of correct digits will be 2 after one, 4 after two, 8 after three, and 16 after four iterations. See the numerical example illustrated. As for the starting procedure, we recommend the following minimization form[6] of some trial values of Newton's corrector

$$S = \min\left(E^*(0), E^*\left(\frac{\pi}{2}\right), E^*(\pi)\right) = \min\left(\frac{M}{1-e}, M + e, \frac{M + \pi e}{1 + e}\right) \tag{8.76}$$

[4] Remark that the Kepler's equation is invariant with respect to the mirror transformation

$$(M, E) \to (-M, -E) \tag{8.71}$$

and the periodic transformation of modulo 2π

$$(M, E) \to (M + 2n\pi, E + 2n\pi) \tag{8.72}$$

and where n is an integer.

[5] The Newton correction completely fails if starting from poor initial guesses

[6] It is easily shown that the minimization form gives an upper bound of the solution. Since the function $f(E)$ is convex in the interval considered, $0 \le E < \pi$, a necessary and sufficient condition for Newton method to converge is to start from an upper bound.

The condition of termination is usually taken as

$$|f(E)| < \delta \tag{8.77}$$

where the tolerance of convergence δ is set as 10^{-14} in double precision computation. As for a more efficient procedure, reader may consult the literature [Fukushima 1997a].

8.3.2 Parabolic Case

Next, consider the parabolic case, $e = 1$. The equation is

$$f_P(\tau) \equiv \tau + \frac{\tau^3}{3} - M_P = 0 \tag{8.78}$$

where τ is the variable to solve for. The equation is called Barker's equation. By symmetry, we can assume that $0 \leq M_P$ and therefore $0 \leq \tau$ without loss of generality. Since Barker's equation is a cubic equation, the exact solution is known[7] as

$$\tau = \frac{2sw}{1+s+s^2} \tag{8.79}$$

where

$$s \equiv \left(\sqrt[3]{w + \sqrt{1+w^2}}\right)^2, \qquad w \equiv \frac{3M_P}{2} \tag{8.80}$$

From the practical point of view, however, we recommend the usage of the Newton method because it is faster. The Newton correction formula is

$$\tau \to \tau^*(\tau) \equiv \tau - \frac{f_P(\tau)}{f_P'(\tau)} = \frac{M_P + (2/3)\tau^3}{1+\tau^2} \tag{8.81}$$

As for the starting procedure, we again recommend the minimization form;

$$S_P = \min\left(\tau^*(0), \tau^*(1), \tau^*(7)\right) = \min\left(M_P, \frac{1}{2}M_P + \frac{1}{3}, \frac{1}{50}M_P + \frac{343}{75}\right) \tag{8.82}$$

As for a more efficient procedure, reader may consult the literature [Fukushima 1998].

[7] See Eq.(4.17) of [Battin 1987]. This Stumpff's rewriting is more robust against the round-off than the original form of Cardano's formula

8.3.3 Hyperbolic Case

Finally, consider the hyperbolic case, $e > 1$. The Kepler's equation is

$$f_H(F) \equiv e \sinh F - F - M_H = 0 \tag{8.83}$$

where F is the variable to solve for. By symmetry, we can assume that $0 \leq M_H$ and therefore $0 \leq F$ without loss of generality. Again, we recommend the usage of the Newton method. The Newton correction formula is

$$F \to F^*(F) \equiv F - \frac{f_H(F)}{f'_H(F)} = \frac{M_H + e(F \cosh F - \sinh F)}{e \cosh F - 1} \tag{8.84}$$

As for the starting procedure, we again recommend the minimization form[8] ;

$$\begin{aligned} S_H &= \min\left(F^*(0), F^*(\ln 2), F^*(8)\right) \\ &= \min\left(\frac{M_H}{e-1}, \frac{M_H + (0.75 - 0.25 \ln 2)e}{1.25e - 1}, \frac{M_H + (8 \cosh 8 - \sinh 8)e}{(\cosh 8)e - 1}\right) \end{aligned} \tag{8.88}$$

As for a more efficient procedure, reader may consult the literature [Fukushima 1997b].

8.4 Differential Operator

An infinitesimal variation of quantities are best described in terms of differential operator d., The followings are the basic properties of the operator;

$$\mathrm{d}\vec{r} = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}, \tag{8.89}$$

$$\mathrm{d}(\alpha x \pm \beta y) = \alpha \mathrm{d}x \pm \beta \mathrm{d}y, \qquad (\alpha, \beta : \text{numerical constants}) \tag{8.90}$$

$$\mathrm{d}(xy) = x\mathrm{d}y + y\mathrm{d}x, \tag{8.91}$$

$$\mathrm{d}\left(x^\nu\right) = \nu x^{\nu-1}\mathrm{d}x, \tag{8.92}$$

[8] The numerical values appeared here are

$$0.75 - 0.25 \ln 2 = 0.5767132048600137 \cdots, \tag{8.85}$$

$$8 \cosh 8 - \sinh 8 = 10433.354464227875 \cdots, \tag{8.86}$$

$$\cosh 8 = 1490.4791612521781 \cdots \tag{8.87}$$

Table 8.2: Example Solution of Kepler's Equation: Elliptic Case

Formula	Numerical Example
Parameters	
(M, e)	(1.0, 0.5)
Corrector	
$E^*(E) \equiv [M + e(\sin E - E\cos E)]/(1 - e\cos E)$	
Starter	
$E^*(0) = M/(1-e)$	2.0
$E^*(\pi/2) = M + e$	1.5
$E^*(\pi) = (M + \pi e)/(1+e)$	1.713864217863264
Newton Correction	
$E_0 = \min(E^*(0), E^*(\pi/2), E^*(\pi))$	$\underline{1.50}$
$E_1 = E^*(E_0)$	$\underline{1.49870}1569636801$
$E_2 = E^*(E_1)$	$\underline{1.4987011335517}897$
$E_3 = E^*(E_2)$	$\underline{1.498701133517848}$

Note: The correct digits are underlined.

Table 8.3: Example Solution of Kepler's Equation: Parabolic Case

Formula	Numerical Example
Parameter	
M_P	0.5
Corrector	
$\tau^*(\tau) \equiv [M_P + (2/3)\tau^3]/[1+\tau^2]$	
Starter	
$\tau^*(0) = M_P$	0.5
$\tau^*(1) = (1/2)M_P + (1/3)$	0.5833333333333333
$\tau^*(7) = (1/50)M_P + (343/75)$	4.583333333333333
Newton Correction	
$\tau_0 = \min(\tau^*(0), \tau^*(1), \tau^*(7))$	$\underline{0.5}$
$\tau_1 = \tau^*(\tau_0)$	$\underline{0.46}66666666666667$
$\tau_2 = \tau^*(\tau_1)$	$\underline{0.466220}6001622060$
$\tau_3 = \tau^*(\tau_2)$	$\underline{0.4662205239107}756$
$\tau_4 = \tau^*(\tau_3)$	$\underline{0.4662205239107734}$

Note: The correct digits are underlined.

Table 8.4: Example Solution of Kepler's Equation: Hyperbolic Case

Formula	Numerical Example
Parameters	
(M_H, e)	(1.0, 2.0)
Corrector	
$F^*(F) \equiv [M_H + (F\cosh F - \sinh F)e]/(e\cosh F - 1)$	
Starter	
$F^*(0) = M_H/(e-1)$	1.0
$F^*(\ln 2) = [M_H + (0.75 - 0.25\ln 2)e]/(1.25e - 1)$	1.435617606480018
$F^*(8) = [M_H + (8\cosh 8 - \sinh 8)e]/[(\cosh 8)e - 1]$	7.002684826450368
Newton Correction	
$F_0 = \min(F^*(0), F^*(\ln 2), F^*(8))$	$\underline{1}.0$
$F_1 = F^*(F_0)$	$\underline{0.8}32034851577094$
$F_2 = F^*(F_1)$	$\underline{0.814}268208303297$
$F_3 = F^*(F_2)$	$\underline{0.8140968}11977352$
$F_4 = F^*(F_3)$	$\underline{0.814096796302133}$

Note: The correct digits are underlined.

Thus

$$\mathrm{d}\left(\vec{r}^2\right) = 2\vec{r}\cdot\mathrm{d}\vec{r} \tag{8.93}$$

and

$$\mathrm{d}\left(\sqrt{x}\right) = \frac{\mathrm{d}x}{2\sqrt{x}}, \tag{8.94}$$

which leads to one of the important rules, that for the magnitude of vector, as

$$\mathrm{d}r = \frac{\vec{r}\cdot\mathrm{d}\vec{r}}{r} \tag{8.95}$$

On the other hand,

$$\mathrm{d}\left(\frac{y}{x}\right) = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2}, \tag{8.96}$$

which leads to another important result, that for the unit vector as

$$\mathrm{d}\left(\frac{\vec{r}}{r}\right) = \left(\frac{\mathrm{d}\vec{r}}{r}\right) - \left[\left(\frac{\vec{r}}{r}\right)\cdot\left(\frac{\mathrm{d}\vec{r}}{r}\right)\right]\left(\frac{\vec{r}}{r}\right), \tag{8.97}$$

Also that for the Newtonian force vector as

$$\mathrm{d}\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^2}\left[\frac{\mathrm{d}\vec{r}}{r} - 3\left\{\left(\frac{\vec{r}}{r}\right)\cdot\left(\frac{\mathrm{d}\vec{r}}{r}\right)\right\}\frac{\vec{r}}{r}\right]. \tag{8.98}$$

which is used in the variational equation of Keplerian orbits. Now, the application to trigonometric functions becomes

$$\mathrm{d}\left(\tan\theta\right) = \left(1+\tan^2\theta\right)\mathrm{d}\theta, \tag{8.99}$$

$$\mathrm{d}\left(\sin\theta\right) = \cos\theta\mathrm{d}\theta, \tag{8.100}$$

and

$$\mathrm{d}\left(\cos\theta\right) = -\sin\theta\mathrm{d}\theta \tag{8.101}$$

By using the last two, we obtain the application to the basic rotation matrices as

$$\mathrm{d}\mathcal{R}_j\left(\theta\right) = \mathcal{Q}_j\left(\theta\right)\mathrm{d}\theta \tag{8.102}$$

Therefore, that for the general rotation matrix becomes

$$\begin{aligned}\mathrm{d}\mathcal{R}_{ijk}\left(\alpha,\beta,\gamma\right) &= \mathcal{Q}_k\left(\gamma\right)\mathcal{R}_j\left(\beta\right)\mathcal{R}_i\left(\alpha\right)\mathrm{d}\gamma \\ &\quad +\mathcal{R}_k\left(\gamma\right)\mathcal{Q}_j\left(\beta\right)\mathcal{R}_i\left(\alpha\right)\mathrm{d}\beta + \mathcal{R}_k\left(\gamma\right)\mathcal{R}_j\left(\beta\right)\mathcal{Q}_i\left(\alpha\right)\mathrm{d}\alpha\end{aligned} \tag{8.103}$$

where $\mathcal{Q}_j$ is the derivatives of basic rotation matrices already given in Eqs (6.67) through (6.69). This becomes simple especially when $\alpha = \beta = \gamma = 0$,

$$\mathrm{d}\mathcal{R}_{ijk}\left(\alpha,\beta,\gamma\right) = \mathcal{Q}_i\left(\alpha\right)\mathrm{d}\alpha + \mathcal{Q}_j\left(\beta\right)\mathrm{d}\beta + \mathcal{Q}_k\left(\gamma\right)\mathrm{d}\gamma \tag{8.104}$$

which is independent on the order of the three angles. On the other hand, those for the inverse trigonometric functions are

$$\mathrm{d}\left(\sin^{-1}s\right) = \frac{\mathrm{d}s}{\sqrt{1-s^2}} \tag{8.105}$$

and

$$\mathrm{d}\left(\cos^{-1}c\right) = \frac{-\mathrm{d}c}{\sqrt{1-c^2}} \tag{8.106}$$

where we assumed that

$$\left|\sin^{-1}s\right| \le \frac{\pi}{2}, \qquad 0 \le \cos^{-1}c \le \pi \tag{8.107}$$

While, that for the arctangent becomes

$$\mathrm{d}\left(\tan^{-1} t\right) = \frac{\mathrm{d}t}{1+t^2}, \tag{8.108}$$

The latter is translated further as

$$\mathrm{d}\left(\tan^{-1} \frac{y}{x}\right) = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}, \tag{8.109}$$

and

$$\mathrm{d}\left(\tan^{-1} \frac{z}{\sqrt{x^2+y^2}}\right) = \frac{\rho^2\mathrm{d}z - z\left(x\mathrm{d}x + y\mathrm{d}y\right)}{\rho r^2} \tag{8.110}$$

where

$$\rho = \sqrt{x^2+y^2}, \qquad r = \sqrt{x^2+y^2+z^2} \tag{8.111}$$

Therefore, the relation for spherical angles (ϕ, λ) connected to a direction vector

$$\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \equiv \begin{pmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{pmatrix} \tag{8.112}$$

becomes

$$\mathrm{d}\lambda = \frac{\cos\lambda\mathrm{d}n_y - \sin\lambda\mathrm{d}n_x}{\cos\phi}, \qquad \mathrm{d}\phi = \cos\phi\mathrm{d}n_z \tag{8.113}$$

8.5 Differential Relations

As an application of the differential operator, we will present some relations.

8.5.1 Spherical Coordinates

The differential relation is expressed in a vector form as

$$\mathrm{d}\vec{r} = (\mathrm{d}r)\,\vec{e}_r + (r\mathrm{d}\phi)\,\vec{e}_\phi + (r\cos\phi\mathrm{d}\lambda)\,\vec{e}_\lambda \tag{8.114}$$

where

$$\vec{e}_r = \begin{pmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{pmatrix}, \qquad \vec{e}_\phi = \begin{pmatrix} -\sin\phi\cos\lambda \\ -\sin\phi\sin\lambda \\ \cos\phi \end{pmatrix}, \qquad \vec{e}_\lambda = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix} \tag{8.115}$$

are the coordinate triad representing a local horizontal reference frame at the position (r, ϕ, λ). Remark that this triad is independent on r.

The above relation is rewritten in componernt-wise as

$$\mathrm{d}x = \cos\phi\cos\lambda\mathrm{d}r - r\sin\phi\cos\lambda\mathrm{d}\phi - r\cos\phi\sin\lambda\mathrm{d}\lambda, \tag{8.116}$$

$$\mathrm{d}y = \cos\phi\sin\lambda\mathrm{d}r - r\sin\phi\sin\lambda\mathrm{d}\phi + r\cos\phi\cos\lambda\mathrm{d}\lambda, \tag{8.117}$$

$$\mathrm{d}z = \sin\phi\mathrm{d}r + r\cos\phi\mathrm{d}\phi \tag{8.118}$$

The inverse relation becomes

$$\mathrm{d}r = \frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{r} = \cos\phi\cos\lambda\mathrm{d}x + \cos\phi\sin\lambda\mathrm{d}y + \sin\phi\mathrm{d}z, \tag{8.119}$$

$$\mathrm{d}\phi = \frac{-z(x\mathrm{d}x + y\mathrm{d}y) + \rho^2\mathrm{d}z}{\rho r^2} = \frac{-\sin\phi\cos\lambda\mathrm{d}x + \sin\phi\sin\lambda\mathrm{d}y + \cos\phi\mathrm{d}z}{r}, \tag{8.120}$$

$$\mathrm{d}\lambda = \frac{-y\mathrm{d}x + x\mathrm{d}y}{\rho^2} = \frac{-\sin\lambda\mathrm{d}x + \cos\lambda\mathrm{d}y}{r\cos\varphi}, \tag{8.121}$$

where

$$\rho = \sqrt{x^2 + y^2} \tag{8.122}$$

8.5.2 Spherical Unit Vectors

As an application of the previous results, we list here the differential relation with respect to unit vectors;

$$\mathrm{d}\vec{n} = \mathrm{d}\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = (\mathrm{d}\phi)\,\vec{e}_\phi + (\cos\phi\mathrm{d}\lambda)\,\vec{e}_\lambda \tag{8.123}$$

Or written in componernt-wise as

$$\mathrm{d}n_x = -\sin\phi\cos\lambda\mathrm{d}\phi - \cos\phi\sin\lambda\mathrm{d}\lambda, \tag{8.124}$$

$$\mathrm{d}n_y = -\sin\phi\sin\lambda\mathrm{d}\phi + \cos\phi\cos\lambda\mathrm{d}\lambda, \tag{8.125}$$

$$\mathrm{d}n_z = \cos\phi\mathrm{d}\phi \tag{8.126}$$

The inverse relation becomes

$$\mathrm{d}\phi = \frac{-n_z(n_x\mathrm{d}n_x + n_y\mathrm{d}n_y) + n_\rho^2\mathrm{d}n_z}{n_\rho}$$

$$= -\sin\phi\cos\lambda \mathrm{d}n_x + \sin\phi\sin\lambda \mathrm{d}n_y + \cos\phi \mathrm{d}n_z, \tag{8.127}$$

$$\mathrm{d}\lambda = \frac{-n_y \mathrm{d}n_x + n_x \mathrm{d}n_y}{n_\rho^2} = \frac{-\sin\lambda \mathrm{d}n_x + \cos\lambda \mathrm{d}n_y}{\cos\varphi}, \tag{8.128}$$

where

$$n_\rho = \sqrt{n_x^2 + n_y^2} \tag{8.129}$$

8.5.3 Spheroidal Coordinates

The differential relation is expressed in a vector form as

$$\mathrm{d}\vec{r} = (\mathrm{d}h)\,\vec{e}_h + (\rho_M\ \mathrm{d}\varphi)\,\vec{e}_\varphi + (\rho_N \cos\varphi\ \mathrm{d}\lambda)\,\vec{e}_\lambda \tag{8.130}$$

where

$$\rho_M = M(\varphi) + h, \qquad \rho_N = N(\varphi) + h, \tag{8.131}$$

$$M(\varphi) = \frac{a\,(1-e^2)}{\left(\sqrt{1-e^2\sin^2\varphi}\right)^3}, \qquad N(\varphi) = \frac{a}{\sqrt{1-e^2\sin^2\varphi}} \tag{8.132}$$

and

$$\vec{e}_h = \begin{pmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{pmatrix}, \qquad \vec{e}_\varphi = \begin{pmatrix} -\sin\varphi\cos\lambda \\ -\sin\varphi\sin\lambda \\ \cos\varphi \end{pmatrix}, \qquad \vec{e}_\lambda = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix} \tag{8.133}$$

are the coordinate triad representing a local horizontal reference frame at the position (φ, λ, h). Remark that this triad is independent on h.

The above relation is rewritten in componernt-wise as

$$\mathrm{d}x = \cos\varphi\cos\lambda \mathrm{d}h - \rho_M \sin\varphi\cos\lambda \mathrm{d}\varphi - \rho_N \cos\varphi\sin\lambda \mathrm{d}\lambda, \tag{8.134}$$

$$\mathrm{d}y = \cos\varphi\sin\lambda \mathrm{d}h - \rho_M \sin\varphi\sin\lambda \mathrm{d}\varphi + \rho_N \cos\varphi\cos\lambda \mathrm{d}\lambda, \tag{8.135}$$

$$\mathrm{d}z = \sin\varphi \mathrm{d}h + \rho_M \cos\varphi \mathrm{d}\varphi \tag{8.136}$$

The inverse relation becomes

$$\mathrm{d}h = \frac{x\mathrm{d}x + y\mathrm{d}y}{\rho} + \frac{z\mathrm{d}z}{\rho_Z} = \cos\varphi\cos\lambda \mathrm{d}x + \cos\varphi\sin\lambda \mathrm{d}y + \sin\varphi \mathrm{d}z, \tag{8.137}$$

$$\mathrm{d}\varphi = \frac{1}{\rho_M}\left(\frac{-z\left(x\mathrm{d}x + y\mathrm{d}y\right)}{\rho\rho_Z} + \frac{\rho\mathrm{d}z}{\rho_N}\right)$$

$$= \frac{-\sin\varphi\cos\lambda\mathrm{d}x + \sin\varphi\sin\lambda\mathrm{d}y + \cos\varphi\mathrm{d}z}{\rho_M}, \tag{8.138}$$

$$\mathrm{d}\lambda = \frac{-y\mathrm{d}x + x\mathrm{d}y}{\rho^2} = \frac{-\sin\lambda\mathrm{d}x + \cos\lambda\mathrm{d}y}{\rho_N\cos\varphi}, \tag{8.139}$$

where

$$\rho = \sqrt{x^2 + y^2} = \rho_N\cos\varphi \tag{8.140}$$

8.6 Difference Operator

A small but finite variation of quantities are best described in terms of difference operator Δ, which is defined as

$$\Delta x = x - x_0. \tag{8.141}$$

The followings are the basic properties of the operator;

$$\Delta\vec{r} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}, \tag{8.142}$$

$$\Delta(\alpha x \pm \beta y) = \alpha\Delta x \pm \beta\Delta y, \qquad (\alpha, \beta : \text{numerical constants}) \tag{8.143}$$

Remark that the rule for multiplication is somewhat different;

$$\Delta(xy) = x\Delta y + y_0\Delta x, \tag{8.144}$$

Note that the above right hand expression is invariant with respect to the exchange of variables, x and y, although it seems to depend on the order of the variables. As its extensions, we obtain

$$\Delta(xyz) = xy\Delta z + xz_0\Delta y + y_0z_0\Delta x, \tag{8.145}$$

$$\Delta\left(x^2\right) = (x + x_0)\Delta x, \tag{8.146}$$

$$\Delta\left(x^3\right) = \left(x^2 + xx_0 + x_0^2\right)\Delta x, \tag{8.147}$$

As a result, we obtain one of the important rules, that for the square of vector, as

$$\Delta\left(\vec{r}^2\right) = (\vec{r} + \vec{r}_0) \cdot \Delta\vec{r} \tag{8.148}$$

On the other hand, Eq.(8.146) is rewritten into

$$\Delta\left(\sqrt{x}\right) = \frac{\Delta x}{\sqrt{x} + \sqrt{x_0}}, \tag{8.149}$$

which leads to another important rule, i.e. that for the magnitude of vector, as

$$\Delta r = \frac{(\vec{r} + \vec{r}_0) \cdot \Delta\vec{r}}{r + r_0}, \qquad \text{where} \qquad r = |\vec{r}|\,. \tag{8.150}$$

The application for the division becomes

$$\Delta\left(\frac{y}{x}\right) = \frac{x_0 \Delta y - y_0 \Delta x}{x x_0}, \tag{8.151}$$

which leads to yet another important result, i.e. that for the unit vector as

$$\Delta\left(\frac{\vec{r}}{r}\right) = \left(\frac{\Delta\vec{r}}{r}\right) - \left[\left(\frac{\vec{r} + \vec{r}_0}{r + r_0}\right) \cdot \left(\frac{\Delta\vec{r}}{r}\right)\right]\left(\frac{\vec{r}_0}{r_0}\right), \tag{8.152}$$

Also that for the Newtonian force vector as

$$\Delta\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^2}\left[\left(\frac{\Delta\vec{r}}{r}\right) - \left(\frac{r^2 + r r_0 + r_0^2}{r_0^2}\right)\left\{\left(\frac{\vec{r} + \vec{r}_0}{r + r_0}\right) \cdot \left(\frac{\Delta\vec{r}}{r}\right)\right\}\left(\frac{\vec{r}_0}{r_0}\right)\right]. \tag{8.153}$$

which is best used in Encke's method.

Now, the application to trigonometric functions becomes

$$\Delta(\tan\theta) = (1 + \tan\theta \tan\theta_0)\tan\Delta\theta, \tag{8.154}$$

$$\Delta(\sin\theta) = 2\cos\left(\frac{\theta + \theta_0}{2}\right)\sin\frac{\Delta\theta}{2}, \tag{8.155}$$

and

$$\Delta(\cos\theta) = -2\sin\left(\frac{\theta + \theta_0}{2}\right)\sin\frac{\Delta\theta}{2}, \tag{8.156}$$

which are easily derived from the addition theorems of trigonometric functions. By using the last two, we obtain the application to the basic rotation matrices as

$$\Delta\mathcal{R}_j(\theta) = 2\sin\left(\frac{\Delta\theta}{2}\right)\mathcal{Q}_j\left(\frac{\theta + \theta_0}{2}\right), \tag{8.157}$$

Therefore, that for the general rotation matrix becomes

$$\Delta\mathcal{R}_{ijk}(\alpha,\beta,\gamma) = 2\sin\left(\frac{\Delta\gamma}{2}\right)\mathcal{Q}_k\left(\frac{\gamma+\gamma_0}{2}\right)\mathcal{R}_j(\beta_0)\,\mathcal{R}_i(\alpha_0)$$

$$+2\sin\left(\frac{\Delta\beta}{2}\right)\mathcal{R}_k(\gamma)\,\mathcal{Q}_j\left(\frac{\beta+\beta_0}{2}\right)\mathcal{R}_i(\alpha_0)$$

$$+2\sin\left(\frac{\Delta\alpha}{2}\right)\mathcal{R}_k(\gamma)\,\mathcal{R}_j(\beta)\,\mathcal{Q}_i\left(\frac{\alpha+\alpha_0}{2}\right), \tag{8.158}$$

where $\mathcal{Q}_j$ is the derivatives of basic rotation matrices already given in Eqs (6.67) through (6.69). Especially, when $\alpha_0 = \beta_0 = \gamma_0 = 0$,

$$\Delta\mathcal{R}_{ijk}(\Delta\alpha,\Delta\beta,\Delta\gamma) = 2\sin\left(\frac{\Delta\gamma}{2}\right)\mathcal{Q}_k\left(\frac{\Delta\gamma}{2}\right) + 2\sin\left(\frac{\Delta\beta}{2}\right)\mathcal{R}_k(\Delta\gamma)\,\mathcal{Q}_j\left(\frac{\Delta\beta}{2}\right)$$

$$+2\sin\left(\frac{\Delta\alpha}{2}\right)\mathcal{R}_k(\Delta\gamma)\,\mathcal{R}_j(\Delta\beta)\,\mathcal{Q}_i\left(\frac{\Delta\alpha}{2}\right), \tag{8.159}$$

As its application, the precession matrix, Eq.(6.37), is rewritten as

$$\mathcal{P} = \mathcal{I} + \Delta\mathcal{P} \tag{8.160}$$

where

$$\Delta\mathcal{P} = -2\sin\left(\frac{z_A}{2}\right)\mathcal{Q}_3\left(\frac{-z_A}{2}\right) + 2\sin\left(\frac{\theta_A}{2}\right)\mathcal{R}_3(-z_A)\,\mathcal{Q}_2\left(\frac{\theta_A}{2}\right)$$

$$-2\sin\left(\frac{\zeta_A}{2}\right)\mathcal{R}_3(-z_A)\,\mathcal{R}_2(\theta_A)\,\mathcal{Q}_3\left(\frac{-\zeta_A}{2}\right) \tag{8.161}$$

On the other hand, the nutation matrix, Eq.(6.42), is rewritten as

$$\mathcal{N} = \mathcal{I} + \Delta\mathcal{N} \tag{8.162}$$

where

$$\Delta\mathcal{N} = -2\sin\left(\frac{\Delta\varepsilon}{2}\right)\mathcal{R}_1(-\varepsilon_A)\,\mathcal{Q}_1\left(-\left(\varepsilon_A + \frac{\Delta\varepsilon}{2}\right)\right)\mathcal{R}_1(\varepsilon_A)$$

$$-2\sin\left(\frac{\Delta\psi}{2}\right)\mathcal{R}_1(-(\varepsilon_A+\Delta\varepsilon))\,\mathcal{Q}_3\left(\frac{-\Delta\psi}{2}\right)\mathcal{R}_1(\varepsilon_A) \tag{8.163}$$

Now, those for the inverse trigonometric functions are somewhat complicated as

$$\Delta\left(\sin^{-1} s\right) = 2\sin^{-1}\left(\frac{\Delta s}{\sqrt{2}\sqrt{c_0^2 - s_0\Delta s + c_0\sqrt{c_0^2 - 2s_0\Delta s - (\Delta s)^2}}}\right) \tag{8.164}$$

and

$$\Delta\left(\cos^{-1} c\right) = 2\sin^{-1}\left(\frac{\Delta c}{\sqrt{2}\sqrt{s_0^2 - c_0\Delta c + s_0\sqrt{s_0^2 - 2c_0\Delta c - (\Delta c)^2}}}\right) \tag{8.165}$$

where

$$c_0 = \sqrt{1 - s_0^2}, \qquad s_0 = \sqrt{1 - c_0^2}, \tag{8.166}$$

While, that for the arctangent becomes

$$\Delta\left(\tan^{-1} t\right) = \tan^{-1}\left(\frac{\Delta t}{1 + tt_0}\right), \tag{8.167}$$

The latter is translated further as

$$\Delta\left(\tan^{-1}\frac{y}{x}\right) = \tan^{-1}\left(\frac{x_0\Delta y - y_0\Delta x}{xx_0 + yy_0}\right), \tag{8.168}$$

and

$$\Delta\left(\tan^{-1}\frac{z}{\sqrt{x^2 + y^2}}\right) = \tan^{-1}\left(\frac{(\rho + \rho_0)\,\rho_0\Delta z - z_0\left((x + x_0)\Delta x + (y + y_0)\Delta y\right)}{(\rho + \rho_0)\left(\rho\rho_0 + zz_0\right)}\right) \tag{8.169}$$

where

$$\rho = \sqrt{x^2 + y^2}, \qquad \rho_0 = \sqrt{x_0^2 + y_0^2} \tag{8.170}$$

8.7 Newton Method

Consider finding the solutions of a one-dimensional nonlinear equation;

$$f(x) = 0 \tag{8.171}$$

Assume that the function $f(x)$ is sufficiently smooth. Also assume that an approximate solution x_0 is already known. Then f is expanded around x_0 as

$$f(x) = f\left(x_0\right) + f'\left(x_0\right)\Delta x + \frac{1}{2}f''\left(x_0\right)\left(\Delta x\right)^2 + \cdots = 0 \tag{8.172}$$

where

$$\Delta x = x - x_0 \tag{8.173}$$

If the second and higher order terms are ignored, an approximate equation with respect to Δx is obtained as

$$f(x_0) + f'(x_0)\,\Delta x = 0 \tag{8.174}$$

This equation is linear and is easily solved as

$$\Delta x = -\frac{f(x_0)}{f'(x_0)} \tag{8.175}$$

This is regarded as a correction to the original approximate solution, x_0. Namely,

$$x_0^* \equiv x^*(x_0) \equiv x_0 + \Delta x = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{8.176}$$

is expected to be a better approximation of the true solution. Thus the iteration

$$x_n = x^*(x_{n-1}), \qquad (n = 1, 2, \cdots) \tag{8.177}$$

will converge to the true solution hopefully. This is the Newton method[9] .

8.7.1 Speed of Convergence

Consider the speed of convergence of Newton method. For the n-th approximate solution x_n, its error is evaluated as

$$x_n - x = x^*(x_{n-1}) - x = (x_{n-1} - x) - \frac{f(x_{n-1})}{f'(x_{n-1})} \tag{8.178}$$

Both $f(x_{n-1})$ and $f'(x_{n-1})$ are expanded around the true solution x as

$$f(x_{n-1}) = f'(x)\,(x_{n-1} - x) + \frac{1}{2} f''(x)\,(x_{n-1} - x)^2 + \cdots \tag{8.179}$$

$$f'(x_{n-1}) = f'(x) + f''(x)\,(x_{n-1} - x) + \cdots \tag{8.180}$$

where we used the relation $f(x) = 0$. Then Eq.(8.178) is approximated as

$$x_n - x = (x_{n-1} - x)$$

[9] Frequently the method had been called as *Newton-Raphson method*. However, the idea itself was invented completely by Newton himself. Thus it is more suitable to call the method by taking the name of Newton himself.

$$-\frac{f'(x)(x_{n-1}-x)+(f''(x)/2)(x_{n-1}-x)^2+(f'''(x)/6)(x_{n-1}-x)^3+\cdots}{f'(x)+f''(x)(x_{n-1}-x)+(f'''(x)/2)(x_{n-1}-x)^2+\cdots}$$

$$=\frac{f''(x)}{2f'(x)}(x_{n-1}-x)^2+\frac{f'''(x)}{3f'(x)}(x_{n-1}-x)^3+\cdots \tag{8.181}$$

Thus, unless $f'(x)=0$,

$$|\Delta x_n| \approx \left|\frac{f''(x)}{2f'(x)}\right| |\Delta x_{n-1}|^2 \tag{8.182}$$

This means that, if $f''(x)/f'(x)$ is of the order of 1, the correct digits will be roughly doubled by each iteration. Namely the Newton method is of the second order as long as the solution is simple.

On the other hand, if $f''(x)=0$, the method is of third order as

$$|\Delta x_n| \approx \left|\frac{f'''(x)}{3f'(x)}\right| |\Delta x_{n-1}|^3 \tag{8.183}$$

8.7.2 Stability

Consider the stability of Newton iteration. First, we assume that the zeros of $f'(x)$ and $f''(x)$ do not coincide with the solution we are looking for, x. Then, if we take a sufficiently small interval containing x, the signatures of $f'(x)$ and $f''(x)$ remain fixed in the interval. By reversing the signatures of x and/or f if necessary, we can assume both of them as plus without loss of generality. In other words, we assume that we can take the interval (x_L, x_U) where

$$x_L < x < x_U, \tag{8.184}$$

namely

$$f(x_L) < f(x) = 0 < f(x_U), \tag{8.185}$$

and

$$f'(x) > 0, \qquad f''(x) > 0, \qquad \text{for} \qquad x_L < x < x_U \tag{8.186}$$

Within this interval, the second derivative of f is always positive, namely f is concave there. In other words, the line being tangent with f at $\tilde{x}$ in $[x_L, x_U]$,

$$g(x) \equiv f(\tilde{x}) + f'(\tilde{x})(x-\tilde{x}) \tag{8.187}$$

is always under the curve of f itself. Namely

$$g(x) < f(x) \qquad \text{for} \qquad x_L < x < x_U \tag{8.188}$$

If we name the zero of $g(x)$ as x^*, then

$$x^* = \tilde{x} - \frac{f(\tilde{x})}{f'(\tilde{x})} < \tilde{x} \tag{8.189}$$

and

$$0 = g(x^*) < f(x^*) \tag{8.190}$$

which implies

$$x < x^* \tag{8.191}$$

Thus

$$x < x^* < \tilde{x} \tag{8.192}$$

Therefore, if

$$x < x_{n-1} < x_U, \tag{8.193}$$

then

$$x < x_n < x_{n-1} \tag{8.194}$$

Namley, for the Newton sequence

$$x_n = x^*(x_{n-1}) \tag{8.195}$$

we can prove that

$$x < \cdots < x_n < x_{n-1} < x_{n-2} < \cdots < x_0 < x_U \tag{8.196}$$

In other words, the Newton sequence is stable in the sense that it is assured that x_n is closer to the true solution than x_{n-1} through the sequence.

8.8 Analytic Solution of Barker's Equation

Consider the solution of a cubic equation named as Barker's equation;

$$\tau + \frac{\tau^3}{3} = M_P \tag{8.197}$$

which is easily rewritten in the second standard form of the cubic equation;

$$\tau^3 + 3p\tau + 2q = 0 \tag{8.198}$$

where

$$p = 1, \qquad q = \frac{-3M_P}{2} \tag{8.199}$$

The determinant D is positive in this case as

$$D = p^3 + q^2 = q^2 + 1 > 0 \tag{8.200}$$

Thus the equation has only one real root, which is given by Cardano's formula as

$$\tau = u - v \tag{8.201}$$

where

$$u = \sqrt[3]{-q + \sqrt{D}}, \qquad v = \sqrt[3]{q + \sqrt{D}} \tag{8.202}$$

Unfortunately, this form is inefficient when q is small, and therefore, when u and v are of the same order of magnitude. In such case, a cancellation in $u - v$ introduces a large round-off error. To avoid this, we may rewrite it as

$$\tau = u - v = \frac{u^3 - v^3}{u^2 + uv + v^2} \tag{8.203}$$

Remark the following relations

$$u^3 - v^3 = \left(-q + \sqrt{D}\right) - \left(q + \sqrt{D}\right) = -2q, \tag{8.204}$$

$$uv = \sqrt[3]{\left(-q + \sqrt{D}\right)\left(q + \sqrt{D}\right)} = \sqrt[3]{\left(\sqrt{D}\right)^2 - q^2} = \sqrt[3]{p^3} = p = 1 \tag{8.205}$$

and therefore

$$v = \frac{1}{u} \tag{8.206}$$

Then the above solution is further rewritten as

$$\tau = \frac{-2q}{u^2 + 1 + u^{-2}} \tag{8.207}$$

If we write

$$w = -q = \frac{3M_p}{2}, \qquad s = u^2 = \left(\sqrt[3]{w + \sqrt{1 + w^2}}\right)^2 \tag{8.208}$$

the final form becomes as

$$\tau = \frac{2sw}{1 + s + s^2} \tag{8.209}$$

This is Stumpff's rewriting [Battin 1987]. Remark that this expression is robust for all nonnegative values of M_P.

8.9 Analytic Solution of Spheroidal Latitude Equation

Consider the solution of an equation appeared in obtaining the spheroidal latitude from the rectangular coordinates;

$$R\tau^4 + U\tau^3 + V\tau - R = 0 \tag{8.210}$$

where we are looking for a real solution in the range

$$0 \le \tau \le 1 \tag{8.211}$$

The conditions on the coefficients are

$$R \ge 0, \qquad V \ge 0, \qquad V > U \tag{8.212}$$

and U can be positive, zero, or negative.

8.9.1 Tribial Case

If $R = 0$, the equation reduces to a cubic equation

$$U\tau^3 + V\tau = \tau(U\tau^2 + V) = 0 \tag{8.213}$$

Clearly $\tau = 0$ is a real solution. If $U < 0$ and $V > 0$, there exist another two real solutions;

$$\tau_{\pm} = \pm\sqrt{\frac{V}{U}} \tag{8.214}$$

However, the condition $V > U$ shows that both are out of the solution interval as

$$\tau_{-} < 0, \qquad 1 < \tau_{+} \tag{8.215}$$

Thus $\tau = 0$ is the solution we sought.

8.9.2 Quartic Equation

Unless $R = 0$, Eq.(8.14) is rewritten in the standard form of the quartic equation as

$$\tau^4 + b\tau^3 + c\tau^2 + d\tau + e = 0 \tag{8.216}$$

where

$$b = \frac{U}{R}, \qquad c = 0, \qquad d = \frac{V}{R}, \qquad e = -1 \tag{8.217}$$

According to Ferrari, the four solutions are two pairs of the solution of the following adjoint quadratic equations;

$$\tau^2 + \frac{b + A_\pm}{2}\tau + \frac{(b + A_\pm)\, y - d}{A_\pm} = 0, \tag{8.218}$$

where

$$A_\pm = \pm\sqrt{8y + b^2 - 4c} = \pm\sqrt{8y + b^2} \tag{8.219}$$

and y is a solution of the following adjoint cubic equation;

$$\begin{aligned} &8y^3 - 4cy^2 + (2bd - 8e)y + e(4c - b^2) - d^2 \\ &\quad = 8y^3 + (2bd + 8)y + \left(b^2 - d^2\right) = 0 \end{aligned} \tag{8.220}$$

The determinants of quadratic equations are

$$B_\pm = \left(\frac{b + A_\pm}{2}\right)^2 - 4\left(\frac{(b + A_\pm)\, y - d}{A_\pm}\right) \tag{8.221}$$

If they are positive, four solutions are given

$$\tau_{\pm,\pm} = -\frac{b + A_\pm}{4} \pm \frac{\sqrt{B_\pm}}{2} \tag{8.222}$$

8.9.3 Reduction of Signatures

Consider the signature which we should select. Imagine the limitting case

$$0 < V = -U \ll 1 \tag{8.223}$$

In this case, the quartic equation reduces to the form

$$\tau^4 - V\tau^3 + V\tau - 1 = (\tau - 1)(\tau + 1)\left(\tau^2 - V\tau + 1\right) = 0 \tag{8.224}$$

Here the solution we look for is $\tau = 1$ since the solution interval is $0 \le \tau \le 1$. The adjoint cubic equation reduces to the form

$$4y^3 + (4 - d^2)y = 0 \tag{8.225}$$

which has only one real solution $y = 0$. Then

$$A_\pm = \pm d \tag{8.226}$$

Judging from the expressions

$$B_\pm = \left(\frac{b + A_\pm}{2}\right)^2 - 4\left(\frac{(b + A_\pm)\, y - d}{A_\pm}\right) \tag{8.227}$$

and

$$\tau_{\pm,\pm} = -\frac{b + A_\pm}{4} \pm \frac{\sqrt{B_\pm}}{2}, \tag{8.228}$$

the condition to lead the solution $\tau = 1$ is to take the minus signature in $A_\pm$ and $B_\pm$ and to take the plus signature in the expression of $\tau_{\pm,\pm}$. Namely the solution we seek for has the form

$$\tau = -\frac{b + A}{4} + \frac{\sqrt{B}}{2} \tag{8.229}$$

where

$$A = -\sqrt{8y + b^2}, \qquad B = \left(\frac{b + A}{2}\right)^2 - 4\left(\frac{(b + A)\, y - d}{A}\right) \tag{8.230}$$

8.9.4 Adjoint Cubic Equation

The adjoint cubic equation is already expressed in the second standard form of the cubic equation as

$$y^3 + 3py + 2q = 0 \tag{8.231}$$

where

$$p = \frac{bd + 4}{12}, \qquad q = \frac{b^2 - d^2}{16} \tag{8.232}$$

If we write

$$S = \frac{U + V}{4}, \qquad T = \frac{V - U}{4} \tag{8.233}$$

then

$$S \geq 0, \qquad T > 0, \qquad U = 2(S - T), \qquad V = 2(S + T), \tag{8.234}$$

and

$$p = \frac{R^2 + S^2 - T^2}{3R^2}, \qquad q = \frac{-2ST}{R^2} \leq 0 \tag{8.235}$$

The form of solution of the cubic equation depends on the signature of the determinant

$$D = p^3 + q^2 \tag{8.236}$$

Since p can be positive, zero, or negative, we should separate the problem into three cases dependeing on the signature of D:

8.9.5 Case $D > 0$

In this case, there exists only one real solution, which is given by Cardano's formula as

$$y = u - v \tag{8.237}$$

where

$$u = \sqrt[3]{-q + \sqrt{D}}, \qquad v = \sqrt[3]{q + \sqrt{D}} \tag{8.238}$$

Again, this form is inefficient when q is small. To avoid a cancellation in $u - v$, we may rewrite it as

$$y = u - v = \frac{u^3 - v^3}{u^2 + uv + v^2} = \frac{-2q}{u^2 + p + (p/u)^2} \tag{8.239}$$

where we used the relation

$$u^3 - v^3 = -2q, \qquad uv = p \tag{8.240}$$

If we write

$$w = -q, \qquad s = u^2 = \left(\sqrt[3]{w + \sqrt{p^3 + w^2}}\right)^2 \tag{8.241}$$

the final form becomes as

$$y = \frac{2sw}{p^2 + ps + s^2} \tag{8.242}$$

8.9.6 Case $D < 0$

In this case, there exist three real roots. They are expressed in terms of cosine function as

$$y_1 = 2r\cos\frac{\theta}{3}, \qquad y_2 = -2r\cos\frac{\pi-\theta}{3}, \qquad y_3 = -2r\cos\frac{\pi+\theta}{3} \tag{8.243}$$

where

$$r = \sqrt{-p}, \qquad \theta = \cos^{-1}\frac{(-q)}{r^3} \tag{8.244}$$

Remark that r is real and positive since

$$p = -\sqrt[3]{(-D)+q^2} < 0 \tag{8.245}$$

Also

$$0 < \frac{(-q)}{r^3} = \sqrt{\frac{q^2}{r^6}} = \sqrt{\frac{q^2}{-p^3}} = \sqrt{\frac{q^2}{(-D)+q^2}} < 1 \tag{8.246}$$

then θ is real and in the range

$$0 < \theta < \frac{\pi}{2} \tag{8.247}$$

Thus

$$0 < \frac{\theta}{3} < \frac{\pi}{6}, \qquad 0 < \frac{\pi-\theta}{3} < \frac{\pi}{6}, \qquad \frac{\pi}{3} < \frac{\pi+\theta}{3} < \frac{\pi}{2}, \tag{8.248}$$

Therefore

$$y_2 < y_3 < 0 < y_1 \tag{8.249}$$

8.9.7 Case $D = 0$

In this case[10] , the equation has a double real root, which we denote y_D, and a simple another real root, y_S. The relations between roots and coefficients become

$$2y_D + y_S = 0, \qquad 2y_D y_S + y_D^2 = 3p, \qquad y_D^2 y_S = -2q \tag{8.250}$$

Thus,

$$y_D = \sqrt[3]{q} = -\sqrt[3]{-q} < 0, \qquad y_S = -2y_D = 2\sqrt[3]{-q} > 0 \tag{8.251}$$

[10] This case is met rarely.

Bibliography

[Battin 1987] Battin, R.H., 1987, An Introduction to the Mathematics and Methods of Astrodynamics, AIAA, New York, NY.

[Colwell 1993] Colwell, P., 1993, Solving Kepler's Equation Over Three Centuries, Willmann-Bell, Richmond, VA.

[Danby 1988] Danby, J.M.A., 1988, Fundamentals of Celestial Mechanics, 2-nd Ed., Willmann-Bell, Richmond, VA.

[Fukushima 1997a] Fukushima, T., 1997a, 'A Method Solving Kepler's Equation without Transcendental Function Evaluation', *Celestial Mechanics and Dynamical Astronomy*, **66**, 309.

[Fukushima 1997b] Fukushima, T., 1997b, 'A Method Solving Kepler's Equation for Hyperbolic Case', *Celestial Mechanics and Dynamical Astronomy*, **68**, 121.

[Fukushima 1998] Fukushima, T., 1998, 'A Fast Procedure Solving Gauss' Form of Kepler's Equation', *Celestial Mechanics and Dynamical Astronomy*, accepted.

[Goldstein 1980] Goldstein, H., 1980, Classical Mechanics, 2-nd Ed., Addison-Wesley Publ. Co., Inc., Redwood City, CA.

[Kovalevsky *et al.* (ed.) 1989] Reference Frames in Astronomy and Geophysics, Kluwer Academic Publ., Dordrecht.

[Murray 1983] Murray, C.A., 1983, Vecgtorial Astrometry, Adam Hilger, Bristol.

[Nagasawa 1981] Nagasawa, K., 1981, Positional Computation of Celestial Objects, *in Japanese*, Chijin-Shokan, Tokyo.

[Seidelmann (ed.) 1992] Explanatory Supplement to the Astronomical Almanac, Univ. Science Books, Mill Valley, CA.

[Smart 1931] Smart, W.M., 1931, Text Book on Spherical Astronomy, Cambridge Univ. Press, London.

[Soffel 1989] Soffel, M.H., 1989, Relativity in Astrometry, Celestial Mechanics and Geodesy, Springer-Verlag, Berlin.

[Taff 1981] Computational Spherical Astronomy, John Wiley and Sons, New York.

[Wolfram 1991] Wolfram S., 1991, Mathematica (2nd ed.), Addison-Wesley Publ. Co., Inc., Redwood City, CA.

[Woolard and Clemence 1966] Woolard, E.W., and Clemence, G.M., 1966, Spherical Astronomy, Academic Press, New York.