A Treatise on Rotation of Celestial Bodies. Part I

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Index (7-11 in Part II)

0. Summary

- Motion of Finite Body
  = Orbital Motion + Rotation
- Basics: Rotation of Rigid Body
  - Orientation: Rotation Matrix
  - Eq. of Motion: Angular Momentum Conservation
  - Solution: Free Rotation + Perturbation
- Topics: Non-Rigidity, Relativity
- Appl.: Earth, Moon, ...

Various Rotation

- Earth: A~B<C, C-A«C
  - Almost Uniform Rotation: UT1
  - Forced Motion of Rot. Axis: Precession, ...
  - Non-Rigidity: Polar Motion, ...
  - Similar Case: Mars
- Moon: A<B<C, B-A<C-A«C
  - Strong Spin-Orbit Coupling
  - Similar Case: Galilean Satellites

Various Rotation (2)

- Asteroids
  - Free Rotation + Small Sun’s Torque
  - Large Non-Sphericity: A<B<C
  - Sometimes Prolate: Ex.) Ida, A<B~C
- Artificial Satellites
  - Arbitrary Shape = Arbitrary (A,B,C)
  - Strong Perturbation: Earth’s Torque, ...
  - Active Control of Attitude

History

- Hipparchus (~BC150): Precession
- Copernicus (1543): Earth's Axis Motion
- Fabricius (1611): Sun's Rotation
- Huygens (1673): Centrifugal Force
- Newton (1687): Law of Motion, Universal Attraction, Explanation of Precession
- Cassini I (1693): Law of Moon's Rotation

History (2)

- D'Alembert (1743): Prediction of Nutation
- Bradley (1747): Discovery of Nutation
- Euler (1765): Eq. of Rigid Body Rotation, Prediction of Polar Motion
- Landen (1775): Landen's Transf.
- Legendre (1786): Elliptic Integrals

History (3)

- Lagrange (1788): Analytical Dynamics
- Poisson (1809): Poisson Approximation
- Jacobi (1829): Elliptic Functions
- Poinsot (1834): Visualization of Rotation
- Hamilton (1834): Canonical Formulation
- Coriolis (1835): Coriolis Force

History (4)

- Serret (1866): Serret Canon. Variables
- Oppolzer (1880): Oppolzer Term
- Chandler (1891): Discovery of Polar Motion
- 1899~: International Latitude Service
- Kimura (1902): Discovery of Z-Term
- Poincare (1910): Fluid Core Rotation
- Einstein (1915): General Relativity

History (5)

- De Sitter (1917): Geodesic Precession
- Andoyer (1923): Canonical Theory of Rotation
- Oort (1927): Galactic Rotation
- Woolard (1953): Modern Nutation Theory
- Molodensky (1961): Nutation of Elastic Earth
- 1962~: Int'l Polar Motion Service

History (6)

- Hori (1966): Pert. Theory by Lie-Transf.
- 1969~: Lunar Laser Ranging Obs.
- Wako (1970): Clarification of Z-Term
- 1976~: Satellite Laser Ranging Obs.
- Lieske et al. (1977): IAU1976 Precession
- Kinoshita (1977): Rigid Earth Nutation Theory based on Hori's Perturbation Theory

History (7)

- Guinot (1979): Non-Rotating Origin
- 1980: MERIT Campaign
- Sasao et al. (1980): SOS Theory
- Wahr (1981): Nutation Theory of Non-rigid Earth
- Seidelmann (1982): IAU1980 Nutation Theory

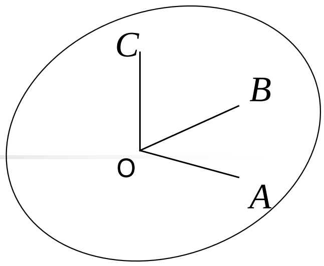
History (8)

- Eckhardt (1981), Moons (1982): Analytical Theory of Lunar Rotation
- Herring (1986): VLBI Determination of Nutation
- Williams (1980‘s): Numerical Theory of Moon’s Rotation, JPL/LE
- 1988~: Int’l Earth Rotation Service

1. Rigid Body

- Idealization of Finite Body
- ~ System of Discrete Mass points
  - Barycenter itself = Orbital Motion
  - Around Barycenter = Rotation
- Simplification $\mathbf{X}(t) = \mathbf{X}_{\mathrm{O}}(t) + \mathbf{R}(t)\mathbf{x}$
  - Newtonian Mechanics
  - Constraint Conditions = Rigidity

Concept



- What is "rigid body"?
  - Set of points with invariant mutual distances
- Description of Position: Basic Triangle
  - Ex.: Center O, Two Points on A- and C-axis
  - Assuming Constraint Force
- Incompatible with Relativity (Why?)
- Idealization in Newton Mechanics (Why?)

Characteristics

- Def.: Body with Internal Invariant Distances
  - Or "System of Discrete Points under Constraints"
- How Many Freedom? (Answer = 6)
  - Ex.: 3 (Barycenter Position) + 2 (Direction of Rotation Axis) + 1 (Rotation Angle)
- Two Coordinate System (CS) are Needed
  - Inertial CS vs Body-Fixed CS
  - Ex.: Geocentric CS vs Earth-Crust-Fixed CS

Number of Freedom = 6

- Outline of Proof
  - Assume that position of basic triangle is given
  - Number of freedom of arbitrary fourth point = 3
  - Distances between the fourth point and the basic triangle are known and fixed
  - 3 unknowns for 3 conditions = fourth point is fixed
  - Arbitrary fourth point is fixed = body is fixed
  - Number of freedom of basic triangle = 3x3 = 9
  - 3 distances between vertices are invariant
  - The rest number of freedom = 9-3 = 6

Expression of Position

- Barycenter (3 positions)
- + Orientation (3 variables)
- Various Ways of Expression
  - Triad of Basis Vectors $\mathbf{E} = (\mathbf{e}_A\ \mathbf{e}_B\ \mathbf{e}_C)$
  - 3 Angles: Eulerian Angles (ψ, θ, ϕ)
  - Rotation Matrix $\mathbf{R} = \mathbf{R}(\psi, \theta, \phi)$
  - Quarternion $q_0 + q_i i + q_j j + q_k k$
  - Spinor $s_0\sigma_0 + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3$

Expression of Motion

- Barycenter Orbital Motion: Linear Velocity
- Rigid Body Rotation: Angular Velocity
- Motion of Barycenter $\mathbf{X}_0(t)$
- Rigid Body Rotation
  - Finite Rotation = Rotation Matrix $\mathbf{R}(t)$
  - Infinitesimal Rotation = Vector Product

$$\mathrm{d}\mathbf{e}_J = \mathbf{e}_J(t+\mathrm{d}t) - \mathbf{e}_J(t) = \boldsymbol{\omega}(t)\,\mathrm{d}t \times \mathbf{e}_J(t)$$

Equation of Motion

- Barycenter: Orbital Motion $\mathbf{p} = m\mathbf{v}$
  - Position **x**, Velocity **v**, Mass m, Momentum **p**
  - Newton: Var. of Mom. = Force **F** $\frac{d\mathbf{p}}{dt} = \mathbf{F}$
- Orientation: Rotation
  - Angle θ, Ang. Velocity ω, Moment of Inertia **I**, Angular Momentum **L** $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$
  - Euler: Var. of Ang. Mom. = Torque **N**
  - Difference: **I** is a Matrix $\frac{d\mathbf{L}}{dt} = \mathbf{N}$

Solution

- Free (=No Torque) Rotation
  - Ang. Vel. (A=B): Simple, Trigonometric F.
  - Ang. Vel. (General): Complicated, Elliptic F.
  - Angle: Incomplete Elliptic Integrals
- Under Torque
  - Generally Torque is Small = Perturbation
  - Analytical Theory vs Numerical Integration
  - Euler Angles vs Canonical Variables

2. Mathematics of Rotation

- Basics of Kinematics of Rigid Body
  - Infinitesimal Rotation: Vector Product
  - Finite Rotation: Rotation Matrix
- Euler's Theorem
- Fundamental Rotation Matrix
- Angular Velocity Vector

Body-Fixed CS

- Two Distinct Coordinate System (CS)
  - Inertial CS (XYZ) vs Body-Fixed CS (ABC)
  - Ex. of Latter CS = Rigid Body CS
- Expression of CS $\mathbf{E} = \begin{pmatrix} \mathbf{e}_A & \mathbf{e}_B & \mathbf{e}_C \end{pmatrix}$
  - Basis Matrix = Triad of Basis Vectors
  - Orthonormality Conditions (6) $\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}$
- Meaning of Vector Components

$$\mathbf{L} = L_X \mathbf{e}_X + L_Y \mathbf{e}_Y + L_Z \mathbf{e}_Z = L_A \mathbf{e}_A + L_B \mathbf{e}_B + L_C \mathbf{e}_C$$

Basis Vectors

- Completeness => Time Var.

$$\frac{\mathrm{d}\mathbf{e}_j}{\mathrm{dt}} = \sum_{k=1}^{3} \Omega_{jk} \mathbf{e}_k$$

- Orthonormality => Anti-symmetry

$$\frac{\mathrm{d}\left(\mathbf{e}_j \cdot \mathbf{e}_k\right)}{\mathrm{dt}} = 0 \rightarrow \frac{\mathrm{d}\mathbf{e}_j}{\mathrm{dt}} \cdot \mathbf{e}_k + \mathbf{e}_j \cdot \frac{\mathrm{d}\mathbf{e}_k}{\mathrm{dt}} = 0 \rightarrow \Omega_{jk} = -\Omega_{kj}$$

- Multiplication of Anti-symmetric Matrix
- = Vector Product

$$\frac{\mathrm{d}\mathbf{e}_A}{\mathrm{dt}} = \boldsymbol{\omega} \times \mathbf{e}_A, \ldots$$

Angular Velocity

- What is Vector ω ?
- Simple Ex.: Const. C-axis Comp. $\boldsymbol{\omega} = \omega \mathbf{e}_C$
  - C-axis is invariant $\mathbf{e}_C(t) = \mathbf{e}_{C0}$
  - Other Axis rotates with constant speed

$$\mathbf{e}_A(t) = \mathbf{e}_{A0} \cos \omega t + \mathbf{e}_{B0} \sin \omega t$$

$$\mathbf{e}_B(t) = -\mathbf{e}_{A0} \sin \omega t + \mathbf{e}_{B0} \cos \omega t$$

- Vector ω = Angular Velocity Vector

Addition Theorem

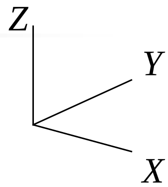
- Add Two Rotations $\frac{d\mathbf{e}_A}{dt} = \boldsymbol{\omega}_1 \times \mathbf{e}_A, \ldots$ $\frac{d\mathbf{e}_A}{dt} = \boldsymbol{\omega}_2 \times \mathbf{e}_A, \ldots$
- Infinitesimal Rotations: δ t is small

$$\mathbf{e}_A^{(1)} \approx \mathbf{e}_A^{(0)} + \boldsymbol{\omega}_1 \delta t \times \mathbf{e}_A^{(0)}$$

$$\mathbf{e}_A^{(2)} \approx \mathbf{e}_A^{(1)} + \boldsymbol{\omega}_2 \delta t \times \mathbf{e}_A^{(1)} \approx \mathbf{e}_A^{(0)} + \left(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2\right) \delta t \times \mathbf{e}_A^{(0)}$$

- Sum of Angular Velocity = Vector Sum

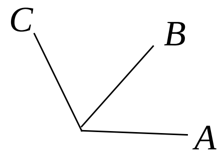
Finite Rotation



- Expressions
  - Matrix, Spinor, Quarternions
- Operation = Matrix Product
  - Then, order-dependent
  - Change Basis M. to Another Basis M.

E(t)

- Rotation Matrix = Orthogonal
  - 9 Comp. but Only 3 Independent



Rigid = Orthogonal

- Rigidity = Distance Invariant

  $$\mathbf{X}^2 = \mathbf{x}^2$$

- Assuming Linear Transf. $\mathbf{X} = \mathbf{R}\mathbf{x}$
- Rigidity = Orthogonality

  $$(\mathbf{R}\mathbf{x})^2 = \mathbf{x}^T\mathbf{R}^T\mathbf{R}\mathbf{x} = \mathbf{x}^2$$

  $$\therefore \mathbf{R}^T\mathbf{R} = \mathbf{1} \text{ or } \mathbf{R}^{-1} = \mathbf{R}^T$$

Angular Velocity M.

- Time Differentiation of Orthogonal M.
  - Orthogonality = Angular Velocity M.

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{1} \;\rightarrow\; \mathbf{R}^{\mathrm{T}}\frac{\mathrm{d}\mathbf{R}}{\mathrm{dt}} = -\left(\frac{\mathrm{d}\mathbf{R}^{\mathrm{T}}}{\mathrm{dt}}\right)\mathbf{R} \equiv -\mathbf{\Omega}$$

- Anti-symmetry

$$\mathbf{\Omega}^{\mathrm{T}} = -\mathbf{\Omega}$$

$$\mathbf{\Omega} = \begin{pmatrix} 0 & -\omega_Z & \omega_Y \\ \omega_Z & 0 & -\omega_X \\ -\omega_Y & \omega_X & 0 \end{pmatrix}$$

Vector Product

- Ang. Vel. M. = 3x3 Anti-Symmetric M.
- Equivalent with 3-Dim. Axial vector

$$\boldsymbol{\Omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \qquad \boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- Anti-symmetric M. = Vector Product

$$\boldsymbol{\Omega}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}$$

Euler's Theorem

- Arbitrary Finite Rotation = Screw Rotation
  = Triple Product of Fund. Rotational M.

$$\mathbf{R} = \mathbf{R}_{ijk}\left(\alpha,\beta,\gamma\right) \equiv \mathbf{R}_k(\gamma)\mathbf{R}_j(\beta)\mathbf{R}_i(\alpha)$$

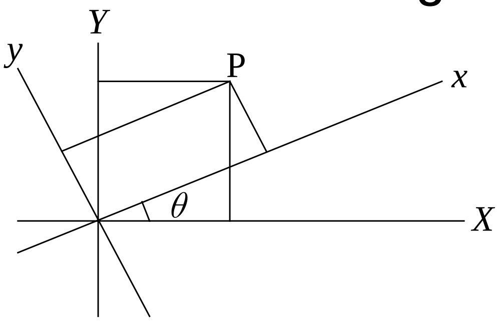
$$\left(\mathbf{R}_{ijk}\left(\alpha,\beta,\gamma\right)\right)^{-1} = \mathbf{R}_{kji}\left(-\gamma,-\beta,-\alpha\right)$$

- Euler Angles = 3 Fund. Rotation Angles
  - Also called as Cardano's angles

Fundamental Rotation

- Ex.: Third Axis Rotation with Angle *q*

$$\mathbf{R}_3(\theta) = \mathbf{R}_Z(\theta)$$



- Coord. Transf.

$$x_P = X_P \cos\theta + Y_P \sin\theta,\ y_P = -X_P \sin\theta + Y_P \cos\theta$$

$$X_P = x_P \cos\theta - y_P \sin\theta,\ Y_P = x_P \sin\theta + y_P \cos\theta$$

Properties of Fund. R.

- Rotation around j-th axis $\mathbf{R}_j(\theta)$
- Inverse = Transpose = Reverse Rotation

$$\left(\mathbf{R}_j(\theta)\right)^{-1} = \left(\mathbf{R}_j(\theta)\right)^{\mathrm{T}} = \mathbf{R}_j(-\theta)$$

- Old Coord. **X**, New Coord. **x**

$$\begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix} = \mathbf{R}_j(\theta) \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix} \qquad \begin{pmatrix} X_P \\ Y_P \\ Z_P \end{pmatrix} = \mathbf{R}_j(-\theta) \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix}$$

Expression of Fund. R.

$$\mathbf{R}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Two Different Rotations

- Two Different Expr. Of Same Vector

$$\mathbf{L} = L_X \mathbf{e}_X + L_Y \mathbf{e}_Y + L_Z \mathbf{e}_Z = L_A \mathbf{e}_A + L_B \mathbf{e}_B + L_C \mathbf{e}_C$$

- CS Rotation vs Rotation of Coordinates

$$\mathbf{E} = \begin{pmatrix} \mathbf{e}_A & \mathbf{e}_B & \mathbf{e}_C \end{pmatrix} \qquad \begin{pmatrix} L_A \\ L_B \\ L_C \end{pmatrix} = \mathbf{R} \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

- What is Relation between **E** and **R**?

Small Angle Approx.

- Assume All Angles are Small

$$\mathbf{R}_3(\theta) \cong \mathbf{1} + \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{1} + \begin{pmatrix} 0 \\ 0 \\ -\theta \end{pmatrix} \times = \mathbf{1} - \theta\, \mathbf{e}_3 \times$$

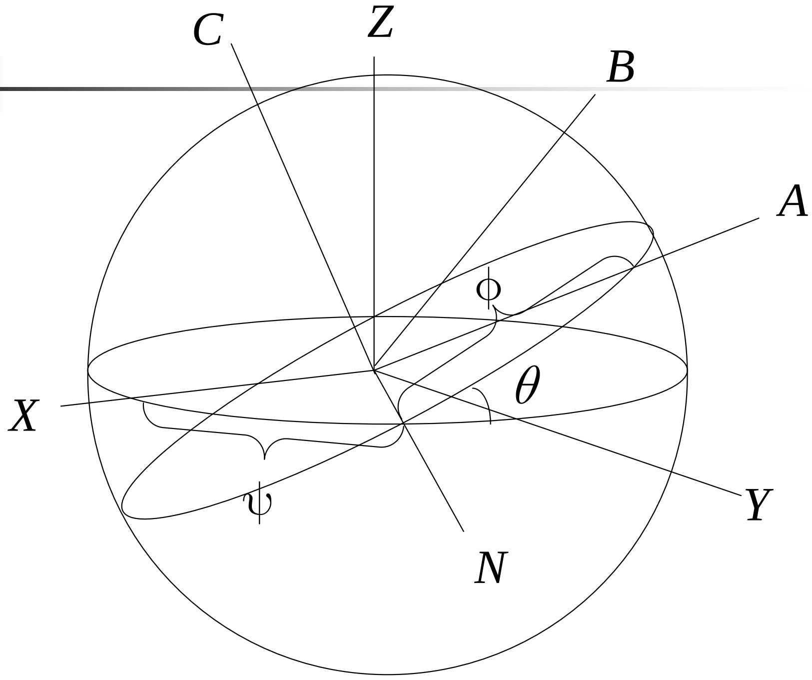
$$\therefore \prod_j \mathbf{R}_j(\theta_j) \cong \mathbf{1} - \left(\sum_j \theta_j \mathbf{e}_j \right) \times$$

Euler Rotation

- Number of Combinations = 3x2x2
  - Why 12 ways?
- 3-1-3 Sequence (= X convention)
  - Classic Dynamics, Continental Style
  - American Style changes ψ and ϕ

$$\mathbf{R}_{313}(\psi,\theta,\phi)=\mathbf{R}_3(\phi)\mathbf{R}_1(\theta)\mathbf{R}_3(\psi)$$

Euler Angles (3-1-3)



3-1-3 Rotation Matrix

- Expression in Inertial CS (Q: Derive)

$$\mathbf{R}_{313}(\psi,\theta,\phi)=\begin{pmatrix} C_\phi C_\psi - S_\phi C_\theta S_\psi & C_\phi S_\psi + S_\phi C_\theta C_\psi & S_\phi S_\theta \\ -S_\phi C_\psi - C_\phi C_\theta S_\psi & -S_\phi S_\psi + C_\phi C_\theta C_\psi & C_\phi S_\theta \\ S_\theta S_\psi & -S_\theta C_\psi & C_\theta \end{pmatrix}$$

$$=\begin{pmatrix} \cos\phi\cos\psi-\sin\phi\cos\theta\sin\psi & \cos\phi\sin\psi+\sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ -\sin\phi\cos\psi-\cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi+\cos\phi\cos\theta\cos\psi & \cos\phi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

3-1-3 Rotation Basis

- Transformation of Basis Vectors

$$(\mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z) \rightarrow (\mathbf{e}_N, \mathbf{e}_M, \mathbf{e}_Z) \rightarrow (\mathbf{e}_N, \mathbf{e}_P, \mathbf{e}_C) \rightarrow (\mathbf{e}_A, \mathbf{e}_B, \mathbf{e}_C)$$

- Expression of Basis Vectors (Q: Derive)

$$\mathbf{e}_N = \mathbf{e}_X \cos\psi + \mathbf{e}_Y \sin\psi, \quad \mathbf{e}_M = -\mathbf{e}_X \sin\psi + \mathbf{e}_Y \cos\psi$$

$$\mathbf{e}_P = \mathbf{e}_M \cos\theta + \mathbf{e}_Z \sin\theta, \quad \mathbf{e}_C = -\mathbf{e}_M \sin\theta + \mathbf{e}_Z \cos\theta$$

$$\mathbf{e}_A = \mathbf{e}_N \cos\phi + \mathbf{e}_P \sin\phi, \quad \mathbf{e}_B = -\mathbf{e}_N \sin\phi + \mathbf{e}_P \cos\phi$$

3-1-3 Rotation Basis (2)

- Relation of Basis and Rotation Matrices

$$\mathbf{E}^{\mathrm{T}} = \begin{pmatrix} \mathbf{e}_A & \mathbf{e}_B & \mathbf{e}_C \end{pmatrix}^{\mathrm{T}} = \mathbf{R}_3(\phi)\mathbf{R}_1(\theta)\mathbf{R}_3(\psi)$$

- Inertial CS Expr. of Basis (Q: Derive)

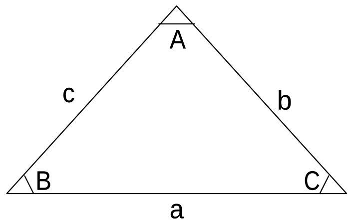
$$\mathbf{e}_N = \begin{pmatrix} \cos\psi \\ \sin\psi \\ 0 \end{pmatrix}, \mathbf{e}_M = \begin{pmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{pmatrix}, \mathbf{e}_P = \begin{pmatrix} -\cos\theta\sin\psi \\ \cos\theta\cos\psi \\ \sin\theta \end{pmatrix}, \mathbf{e}_C = \begin{pmatrix} \sin\theta\sin\psi \\ -\sin\theta\cos\psi \\ \cos\theta \end{pmatrix}$$

$$\mathbf{e}_A = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi \\ \cos\phi\sin\psi + \sin\phi\cos\theta\cos\psi \\ \sin\phi\sin\theta \end{pmatrix}, \mathbf{e}_B = \begin{pmatrix} -\sin\phi\cos\psi - \cos\phi\cos\theta\sin\psi \\ -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi \\ \cos\phi\sin\theta \end{pmatrix}$$

Spherical Triangles

- Angles: A,B,C + Arcs: a,b,c
- 2 Expr. Of Same Rotation

$$\mathbf{R}_1(A)\mathbf{R}_3(c)\mathbf{R}_1(B)=\mathbf{R}_3(b)\mathbf{R}_1(\pi-C)\mathbf{R}_3(a)$$

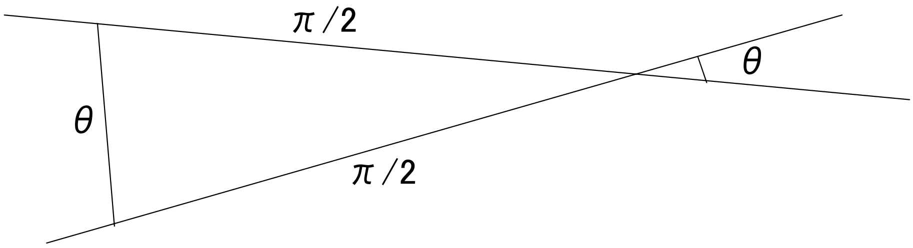


- Formulas of Spherical Triangles
  - Sine $\sin a \sin B = \sin b \sin A$
  - 1st Cosine $\cos a = \cos b \cos c + \sin b \sin c \cos A$
  - 2nd Cosine $\cos A = -\cos B \cos C + \sin B \sin C \cos a$

Defect of 3-1-3 Seq.

- Difficulty in Expressing Axis-2 Rotation

$$\mathbf{R}_2(\theta) = \mathbf{R}_3\left(\frac{-\pi}{2}\right)\mathbf{R}_1(\theta)\mathbf{R}_3\left(\frac{\pi}{2}\right)$$



Defects of 3-1-3 Seq. (2)

- Degeneracy in case of small angles

$$\mathbf{R}_{313}(\psi,\theta,\phi) \cong \mathbf{1} - \begin{pmatrix} \theta \\ 0 \\ \phi+\psi \end{pmatrix} \times$$

- Cause: 2 indices are the same
- Solution
  - All-different-indices Seq. like 1-2-3 Seq.

3-2-3 Sequence

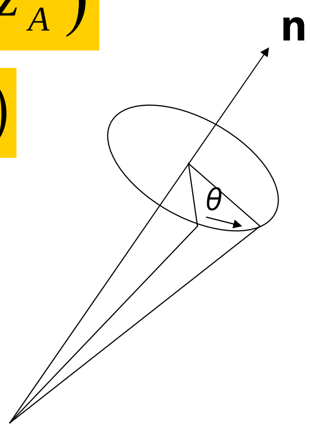
- = Y Convention = English Style

$$\mathbf{P} = \mathbf{R}_{323}\left(-\zeta_A, \theta_A, -z_A\right)$$

- Screw Rotation $\mathbf{R}_{323}\left(\lambda, \varphi, \theta\right)$

= Rotation around Axis **n**

$$\mathbf{n} = \begin{pmatrix} \sin\varphi\cos\lambda \\ \sin\varphi\sin\lambda \\ \cos\varphi \end{pmatrix}$$



Screw Rotation

- Screw Rotation
  - = Rodriguez's Rotation Matrix

$$\mathbf{X} = \mathrm{R}_{323}\left(\lambda,\varphi,\theta\right)\mathbf{x}$$
$$= \mathbf{x} + \sin\theta\,\mathbf{n}\times\mathbf{x} + \left(1-\cos\theta\right)\mathbf{n}\times\left(\mathbf{n}\times\mathbf{x}\right)$$

- Angular Velocity = Constant $\boldsymbol{\omega} = \omega\mathbf{n}$
  - Then, Rotation Angle is $\theta = \omega t$

Other Sequences

- 1-3-1 Seq.: used in Nutation Matrix

$$\mathbf{N} = \mathbf{R}_{131}\left(\varepsilon_A, -\Delta\psi, -\left(\varepsilon_A + \Delta\varepsilon\right)\right)$$

- 2-1-3 Seq.: Polar Motion + Sidereal R.

$$\mathbf{W}\,\mathbf{S} = \mathbf{R}_{312}\left(\Theta, -y_p, -x_p\right)$$

1-2-3 Sequence

- Aerodynamics, Attitude Dynamics
  - Mathematically Best

$$\mathbf{R}_{123}\left(\alpha,\beta,\gamma\right)=\mathbf{R}_3(\gamma)\mathbf{R}_2(\beta)\mathbf{R}_1(\alpha)$$

$$=\begin{pmatrix} C_\gamma C_\beta & C_\gamma S_\beta S_\alpha + S_\gamma C_\alpha & -C_\gamma S_\beta C_\alpha + S_\gamma S_\alpha \\ -S_\gamma C_\beta & -S_\gamma S_\beta S_\alpha + C_\gamma C_\alpha & S_\gamma S_\beta C_\alpha + C_\gamma S_\alpha \\ S_\beta & -C_\beta S_\alpha & C_\beta C_\alpha \end{pmatrix} \cong \mathbf{I} - \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}\times$$

Angular Velocity Expr.

- Vector Expression of Infinitesimal Rotation

$$\boldsymbol{\omega}\mathrm{dt} = \mathrm{d}\boldsymbol{\theta} = \sum_j \mathbf{e}_{\theta_j} \mathrm{d}\theta_j \rightarrow \boldsymbol{\omega} = \sum_j \left(\frac{\mathrm{d}\theta_j}{\mathrm{dt}} \right) \mathbf{e}_{\theta_j}$$

- Rotation Axis = Direction of Angular Velocity
  - Note 1: Ang. Velocity ≠ Time Var. of Angles
  - Note 2: Basis Vector Change with Rotation
  - Note 3: Angular Velocity is independent on the order of Rotational Operations

An Example of Expr.

- 3-1-3 Seq. $\boldsymbol{\omega} = \frac{\mathrm{d}\psi}{\mathrm{dt}}\mathbf{e}_\psi + \frac{\mathrm{d}\theta}{\mathrm{dt}}\mathbf{e}_\theta + \frac{\mathrm{d}\phi}{\mathrm{dt}}\mathbf{e}_\phi$
- Inertial CS Repr. $\mathbf{e}_\psi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{e}_\theta = \begin{pmatrix} \cos\psi \\ \sin\psi \\ 0 \end{pmatrix}, \mathbf{e}_\phi = \begin{pmatrix} \sin\theta\sin\psi \\ -\sin\theta\cos\psi \\ \cos\theta \end{pmatrix}$
- Components

$$\begin{pmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{pmatrix} = \begin{pmatrix} \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi \\ \dot\theta\sin\psi - \dot\phi\sin\theta\cos\psi \\ \dot\psi + \dot\phi\cos\theta \end{pmatrix}$$

Another Example

- Body-Fixed CS Repr.

$$\mathbf{e}_\psi = \begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix}, \mathbf{e}_\theta = \begin{pmatrix} \cos\phi \\ -\sin\phi \\ 0 \end{pmatrix}, \mathbf{e}_\phi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Components
- = Euler's Kinematic Eq. of Rotation

$$\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix} = \begin{pmatrix} \dot\psi\sin\theta\sin\phi + \dot\theta\cos\phi \\ \dot\psi\sin\theta\cos\phi - \dot\theta\sin\phi \\ \dot\psi\cos\theta + \dot\phi \end{pmatrix}$$

Velocity Transf.

- Variation of Basis Vector

$$\frac{\mathrm{d}\mathbf{e}_A}{\mathrm{dt}} = \boldsymbol{\omega} \times \mathbf{e}_A,\ \ldots$$

- Transf. Law in Body-Fixed CS
- D/Dt = Diff. in Body-Fixed CS

$$\mathbf{v} = \frac{\mathrm{D}\mathbf{x}}{\mathrm{Dt}} + \boldsymbol{\omega} \times \mathbf{x}$$

  - Proof

$$\mathbf{X} = x_A\mathbf{e}_A + x_B\mathbf{e}_B + x_C\mathbf{e}_C,\ \mathbf{V} = v_A\mathbf{e}_A + v_B\mathbf{e}_B + v_C\mathbf{e}_C$$

$$\mathbf{x} = \left(x_A, x_B, x_C\right)^{\mathrm{T}},\ \mathbf{v} = \left(v_A, v_B, v_C\right)^{\mathrm{T}}$$

$$\mathbf{V} = \frac{\mathrm{d}\mathbf{X}}{\mathrm{dt}} = \frac{\mathrm{d}x_A}{\mathrm{dt}}\mathbf{e}_A + \frac{\mathrm{d}x_B}{\mathrm{dt}}\mathbf{e}_B + \frac{\mathrm{d}x_C}{\mathrm{dt}}\mathbf{e}_C + x_A\frac{\mathrm{d}\mathbf{e}_A}{\mathrm{dt}} + x_B\frac{\mathrm{d}\mathbf{e}_B}{\mathrm{dt}} + x_C\frac{\mathrm{d}\mathbf{e}_C}{\mathrm{dt}}$$

$$\therefore v_A = \frac{\mathrm{d}x_A}{\mathrm{dt}} + \left(\omega_B x_C - \omega_C x_B\right) \rightarrow \mathbf{v} = \frac{\mathrm{D}\mathbf{x}}{\mathrm{Dt}} + \boldsymbol{\omega} \times \mathbf{x}$$

Another Expr. of Transf.

- Symbolic Form
- Two Time Diff.
  - True Time Diff. in Inertial CS: d/dt
  - Component in Body-Fixed CS : D/Dt
- Inertial CS vs Body-Fixed CS

$$\frac{d}{dt} = \frac{D}{Dt} + \boldsymbol{\omega} \times$$

$$\mathbf{V} = \frac{d\mathbf{X}}{dt}$$

$$\mathbf{v} = \frac{D\mathbf{x}}{Dt} + \boldsymbol{\omega} \times \mathbf{x}$$

3. Physics of Rotation

- Total Momentum & Total Angular Momentum
- Conservation of Angular Momentum
- Ang. Mom. Expr. in Body-Fixed CS
- Moment of Inertia
- Expression of Torque
- Euler's Dynamical Eq. of Rotation

Newton's Eq. of Motion

- For Each Component of Finite Body
- Total Force = External + Internal
- Internal Force: Molecular, Atomic, EM,...

$$m_K \frac{d^2 \mathbf{X}_K}{dt^2} = \mathbf{F}_K + \sum_{J \neq K} \mathbf{F}_{JK}$$

Motion of Barycenter

- Total Mass $M \equiv \sum_K m_K$
- Barycenter $M\mathbf{X}_{\mathrm{O}} \equiv \sum_K m_K \mathbf{X}_K$
- Internal F.: Weak Law of Reaction $\mathbf{F}_{JK} = -\mathbf{F}_{KJ}$
- Barycenter Mot. independent on Internal F.

$$M\frac{\mathrm{d}^2\mathbf{X}_{\mathrm{O}}}{\mathrm{dt}^2} = \mathbf{F} \equiv \sum_K \mathbf{F}_K$$

Angular Momentum

- Linear Mom. vs Ang. Mom. $\mathbf{L}_K \equiv m_K \mathbf{X}_K \times \mathbf{V}_K$
- Total Ang. Mom. $\mathbf{L}_{\text{Total}} \equiv \sum_K \mathbf{L}_K$

= Orbital AM + Rotational AM

  - Q: Derive

$$\mathbf{L}_{\text{Total}} = \mathbf{L}_{\text{O}} + \mathbf{L},\ \mathbf{L}_{\text{O}} = M\mathbf{X}_{\text{O}} \times \mathbf{V}_{\text{O}},$$

$$\mathbf{L} = \sum_K m_K \left(\mathbf{X}_K - \mathbf{X}_{\text{O}}\right) \times \left(\mathbf{V}_K - \mathbf{V}_{\text{O}}\right)$$

Discrete to Continuous

- Discrete Sum → Continuous Integration
- Mass Density Distribution Function $\rho(\mathbf{x})$

$$\sum_K m_K \mathbf{P}_K \to \int \mathbf{P}(\mathbf{x})\rho(\mathbf{x})\mathrm{d}^3\mathbf{x}$$

- Ex.: Rotational Angular Momentum, **L**

$$\mathbf{L} = \sum_K m_K \left(\mathbf{X}_K - \mathbf{X}_{\mathrm{O}}\right)\times\left(\mathbf{V}_K - \mathbf{V}_{\mathrm{O}}\right)$$

$$\to \mathbf{L} = \int \mathbf{x}\times\mathbf{v}(\mathbf{x})\rho(\mathbf{x})\mathrm{d}^3\mathbf{x}$$

Total Angular Mom.

- Strong Law of Reaction
  - = Constraint F. is Central (Invalid for EM force)

$$\mathbf{F}_{JK} \propto \mathbf{X}_J - \mathbf{X}_K$$

- Time Var. of Total AM by External Force
  - Q: Show

$$\frac{d\mathbf{L}_{\mathrm{Total}}}{dt} = \sum_K \mathbf{X}_K \times \mathbf{F}_K$$

Rotational Angular Mom.

- Accelerations $\mathbf{A}_{\mathrm{O}} \equiv \frac{\mathbf{F}}{M}$ $\mathbf{A}_K \equiv \frac{\mathbf{F}_K}{m_K}$
- Barycenter O
- Time Var. of Rot. Ang. Mom. by Torque

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{dt}} = \sum_K m_K \left(\mathbf{X}_K - \mathbf{X}_{\mathrm{O}}\right) \times \left(\mathbf{A}_K - \mathbf{A}_{\mathrm{O}}\right)$$

$$= \int \mathbf{x} \times \mathbf{a}(\mathbf{x}) \rho(\mathbf{x}) \mathrm{d}^3\mathbf{x} \equiv \mathbf{N}$$

- Tidal Acceleration $\mathbf{a}_K = \mathbf{A}_K - \mathbf{A}_{\mathrm{O}}$

Tidal Acceleration

- Def: Difference from Barycenter's Accel., $\mathbf{A}_O$
- Usual Approximation
  - Depend on Position Only
  - Up to 1st Order of Variation

$$\mathbf{a}_K \approx \left(\frac{\partial \mathbf{A}_K}{\partial \mathbf{x}_K}\right)_O \mathbf{x}_K$$

- Ex.: Gravitational F. by External Mass point

$$\mathbf{A}_K = \frac{Gm}{\left|\mathbf{r}-\mathbf{x}_K\right|^3}\left(\mathbf{r}-\mathbf{x}_K\right)$$

$$\mathbf{a}_K = \left(\frac{Gm}{r^5}\right)\left[3\left(\mathbf{r}\bullet\mathbf{x}_K\right)\mathbf{r} - r^2\mathbf{x}_K\right]$$

- Note: Not always $\mathbf{A}_O=\mathbf{A}_K(\mathbf{x}_K=\mathbf{x}_O)$

Rotational Ang. Mom. (2)

- Rigid Body Approx.: D**x**/Dt=0→**v**=$\omega\times$**x**
  - Rot. AM, **L**, is Linear Function of ω

$$\mathbf{L} = \int \mathbf{x}\times\left(\boldsymbol{\omega}\times\mathbf{x}\right)\rho\left(\mathbf{x}\right)\mathrm{d}^3\mathbf{x} = \mathbf{I}\boldsymbol{\omega}$$

  - Moment-of-Inertia Tensor **I** (Q: Show)

$$\left(\mathbf{I}\right)_{ij} = I_{ij} = \int\left(\mathbf{x}^2\delta_{ij} - x_i x_j\right)\rho\left(\mathbf{x}\right)\mathrm{d}^3\mathbf{x}$$

  - “AM Axis” = Direction of AM Vector

Moment of Inertia

- Mass Moment Tensor

$$J_{ij} = \int x_i x_j \rho(\mathbf{x}) \, \mathrm{d}^3\mathbf{x}$$

- Moment-of-Inertia Tensor (Q: Show)

$$I_{XX} = \frac{1}{2}\left(J_{YY} + J_{ZZ} - J_{XX}\right) \qquad I_{ij} = -J_{ij} \quad (i \neq j)$$

  - Cyclic Change of Indices（X→Y→Z）

Rotational Energy

- Again, Rigid Body Approx.: $\mathbf{v} = \omega \times \mathbf{x}$

$$T = \frac{1}{2}\int \mathbf{v}^2 \rho(\mathbf{x})\,\mathrm{d}^3\mathbf{x} = \frac{1}{2}\int (\boldsymbol{\omega}\times\mathbf{x})^2 \rho(\mathbf{x})\,\mathrm{d}^3\mathbf{x}$$

$$= \frac{\boldsymbol{\omega}\cdot\mathbf{L}}{2} = \frac{\boldsymbol{\omega}\cdot(\mathbf{I}\boldsymbol{\omega})}{2} = \frac{1}{2}\sum_{i,j=1}^{3} I_{ij}\omega_i\omega_j$$

- Maximum Energy for Constant AM
  → "Rotation Axis" = "AM Axis"

Ellipsoid of Inertia

- Normal Ellipsoid assoc. with **I** $\sum_{i,j=1}^{3} I_{ij} x_i x_j = 1$
- Quadratic Form: Diagonalization

  $$\tilde{I}_{11}\tilde{x}_1^2 + \tilde{I}_{22}\tilde{x}_2^2 + \tilde{I}_{33}\tilde{x}_3^2 = 1$$

- Diag. Elem. = Principal Mom. Of Inertia

  $$A \le B \le C \qquad Ax^2 + By^2 + Cz^2 = 1$$

- Inverse $a \equiv A^{-1} \ge b \equiv B^{-1} \ge c \equiv C^{-1}$

Principal CS

- = Principal Moment-of-Inertia CS
- CS where **I** is Diagonalized
- Well-used as a Body-Fixed CS
- Expr. in Principal CS
- Figure Axis

=C-Axis of Principal CS

$$\mathbf{I} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

Expr. In Principal CS

- Rotational Quant. Are Plainly Expressed
- Rotational AM

$$\begin{pmatrix} L_A \\ L_B \\ L_C \end{pmatrix} = \begin{pmatrix} A\omega_A \\ B\omega_B \\ C\omega_C \end{pmatrix}$$

- Rotational Energy

$$T = \frac{1}{2}\left(A\omega_A^2 + B\omega_B^2 + C\omega_C^2 \right)$$

Triad of Principal CS

- In Inertial CS: 3-1-3 Euler Angles

$$(\mathbf{e}_A \quad \mathbf{e}_B \quad \mathbf{e}_C)^{\mathrm{T}} = \mathbf{R}_3(\phi)\mathbf{R}_1(\theta)\mathbf{R}_3(\psi)$$

$$\mathbf{e}_A = \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi \\ \sin\psi\cos\phi + \cos\psi\cos\theta\sin\phi \\ \sin\theta\sin\phi \end{pmatrix}$$

$$\mathbf{e}_B = \begin{pmatrix} -\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi \\ -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi \\ \sin\theta\cos\phi \end{pmatrix}$$

$$\mathbf{e}_C = \begin{pmatrix} \sin\psi\sin\theta \\ -\cos\psi\sin\theta \\ \cos\theta \end{pmatrix}$$

Moment of Inertia (2)

- Dyadic Expr. by Triad of Principal CS

$$\mathbf{I} = A\mathbf{e}_A \otimes \mathbf{e}_A + B\mathbf{e}_B \otimes \mathbf{e}_B + C\mathbf{e}_C \otimes \mathbf{e}_C$$

- Easy Inverse Matrix Repr. (Q: Show)

$$\mathbf{I}^{-1} = a\mathbf{e}_A \otimes \mathbf{e}_A + b\mathbf{e}_B \otimes \mathbf{e}_B + c\mathbf{e}_C \otimes \mathbf{e}_C$$

- Dyadic = Direct Product of Vector

$$(\mathbf{a} \otimes \mathbf{b})_{jk} = (\mathbf{a})_j (\mathbf{b})_k = a_j b_k$$

Principal Mom. Inertia

- Spherically Symmetric
  - A=B=C
- Rot. Symm. (Oblate or Hamburger-like)
  - A=B<C
- Rot. Symm. (Prolate or Lemon-shaped)
  - A<B=C
- General: A<B<C

Principal M. Inertia (2)

- Uniform Triaxial Ellipsoid (Q: Confirm)
  - Cyclic Change of Indices (A→B→C)

$$A = \frac{4\pi}{15} R_A R_B R_C \left(R_B^2 + R_C^2 \right)$$

$$\frac{C-A}{B} = \frac{R_A^2 - R_C^2}{R_A^2 + R_C^2}$$

- Ex: Asteroid Ida, A<B~C

$$R_A = 59.8 \text{ km}, R_B = 25.4 \text{ km}, R_C = 18.6 \text{ km}$$

$$\frac{A}{C} = 0.235, \frac{B}{C} = 0.929$$

Principal M. Inertia (3)

- Earth: A~B<C

$$\frac{C_E - A_E}{C_E} = 3.284741\times10^{-3}, \frac{B_E - A_E}{C_E} = 2.196\times10^{-6}, \frac{C_E}{M_E R_E^2} = 0.330701$$

$$GM_E = 3.986005\times10^{14}\,\mathrm{m^3s^{-2}}, R_E = 6.3781366\times10^{6}\,\mathrm{m}$$

- Moon: A<B<C

$$\frac{C_M - A_M}{B_M} = 6.32\times10^{-4}, \frac{B_M - A_M}{C_M} = 2.33\times10^{-4}, \frac{C_M}{M_M R_M^2} = 0.389$$

$$\frac{M_M}{M_E} = 1.230002\times10^{-2}, R_M = 1.738\times10^{6}\,\mathrm{m}$$

Three Axes

- Rotat. ω vs Ang.Mom. **L** vs Figure $\mathbf{e}_C$
  - Confusing
  - Which is so-called “Pole”?
- Expressions in Principal CS

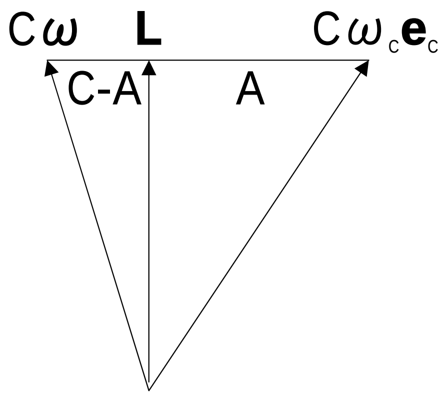
$$\boldsymbol{\omega} = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}, \ \mathbf{L} = \begin{pmatrix} A\omega_A \\ B\omega_B \\ C\omega_C \end{pmatrix}, \ \mathbf{e}_C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Three Axes (2)

- If A=B<C
  - Q: Show

$$\mathbf{L} = A\boldsymbol{\omega} + \left(C - A\right)\omega_C \mathbf{e}_C$$

- 3 Axes are on the Same Plane
- If A~C, AM Axis is close to Rot. Axis but far from Figure Axis



Visualization of Rotation

- Method of Poinsot (1851)
  - Free Rotation: Moving and Fixed Cones
- Polhode
  - Orbit of Rot. Axis on Ellipsoid of Inertia

$$\omega_A(t), \omega_B(t), \omega_C(t) \qquad A\omega_A^2 + B\omega_B^2 + C\omega_C^2 = 2T$$

- Herpolhode
  - Orbit of Rot. Axis on Plane in Inertial CS

$$\omega_X(t), \omega_Y(t), \omega_Z(t) \qquad c_X\omega_X + c_Y\omega_Y + c_Z\omega_Z = c_0$$

Eq. of Rotation

- Conservation of Rotational Ang. Mom.
  - Inertial CS + Ang. Mom. $\frac{d\mathbf{L}}{dt} = \mathbf{N}$
  - Principal CS + Ang. Mom.
    $$\frac{D\mathbf{L}}{Dt} = \mathbf{N} - \left(\mathbf{I}^{-1}\mathbf{L}\right) \times \mathbf{L}$$
  - Principal CS + Ang. Vel. (Euler 1765)
    $$\frac{D\boldsymbol{\omega}}{Dt} = \mathbf{I}^{-1}\left(\mathbf{N} - \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega}\right)$$

Tidal Torque

- Main Term of External Torque

$$\mathbf{N} = \frac{3\mu}{r^5}\int(\mathbf{r}\bullet\mathbf{x})(\mathbf{r}\times\mathbf{x})\rho(\mathbf{x})\,\mathrm{d}^3\mathbf{x}$$

- Ex.: Moon+Sun Acting on Earth
  - Moon (or Sun)'s GM = μ 、Geocentric Position **r**
- General Torque: from Law of Reaction
  - F = Force by Body on External Point

$$\mathbf{N} = -\mathbf{r}\times\mathbf{F}$$

Eq. of Rotation (2)

- Inertial CS $\left(\psi, \theta, \phi, \mathbf{L}\right)$
  - Kinematic Part = Expr. of Ang. Vel. with Time Var. of Euler Angles

$$\frac{\mathrm{d}\psi}{\mathrm{dt}} = \omega_Z - \left(\frac{\mathrm{d}\phi}{\mathrm{dt}}\right)\cos\theta$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}} = \omega_X \cos\psi + \omega_Y \sin\psi$$

$$\frac{\mathrm{d}\phi}{\mathrm{dt}} = \frac{\omega_X \sin\psi - \omega_Y \cos\psi}{\sin\theta}$$

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{dt}} = \mathbf{N}$$

$$\boldsymbol{\omega} = \mathbf{I}^{-1}\mathbf{L}$$

$$\mathbf{N} = N_A \mathbf{e}_A + N_B \mathbf{e}_B + N_C \mathbf{e}_C$$

Eq. of Rotation (3)

- Principal CS, Ang. Mom. $(\psi, \theta, \phi, L_A, L_B, L_C)$
  - Q: Show

$$\frac{\mathrm{d}\psi}{\mathrm{dt}} = \frac{aL_A \sin\phi + bL_B \cos\phi}{\sin\theta}$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}} = aL_A \cos\phi - bL_B \sin\phi$$

$$\frac{\mathrm{d}\phi}{\mathrm{dt}} = cL_C - \left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)\cos\theta$$

$$\frac{\mathrm{d}L_A}{\mathrm{dt}} = N_A - (b-c)L_B L_C$$

$$\frac{\mathrm{d}L_B}{\mathrm{dt}} = N_B - (c-a)L_C L_A$$

$$\frac{\mathrm{d}L_C}{\mathrm{dt}} = N_C - (a-b)L_A L_B$$

Eq. of Rotation (4)

- Principal CS, Ang. Vel. $\left(\psi,\theta,\phi,\omega_A,\omega_B,\omega_C\right)$
  - Q: Show

$$\frac{\mathrm{d}\psi}{\mathrm{dt}}=\frac{\omega_A\sin\phi+\omega_B\cos\phi}{\sin\theta}$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}}=\omega_A\cos\phi-\omega_B\sin\phi$$

$$\frac{\mathrm{d}\phi}{\mathrm{dt}}=\omega_C-\left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)\cos\theta$$

$$\frac{\mathrm{d}\omega_A}{\mathrm{dt}}=\frac{N_A-\left(C-B\right)\omega_B\omega_C}{A}$$

$$\frac{\mathrm{d}\omega_B}{\mathrm{dt}}=\frac{N_B-\left(A-C\right)\omega_C\omega_A}{B}$$

$$\frac{\mathrm{d}\omega_C}{\mathrm{dt}}=\frac{N_C-\left(B-A\right)\omega_A\omega_B}{C}$$

Trivial Solution: A=B=C

- Simplest Solution: Spherically Symmetric
- Zero Torque (Q: Why?)
- → Const. Ang. Mom. → Const. Ang. Vel.
- A=B=C → AM Axis = Rot. Axis
- Coord. Transf.: New C-Axis = AM Axis
- Time Variation of Angles
  - Rot. Angle is Linear, Others =Const.
  - Result: Uniform Rotation

Orbit vs Rotation

Item	Orbit	Rotation
Position	**X**	Euler A., $(\psi\ \theta\ \phi)$
Velocity	**V**	Ang. Vel., $\boldsymbol{\omega}$
Mass	M	Mom. Inertia, **I**
Momentum	**P**=M**V**	**L**=**I**ω
Kin. Energy	M$\mathbf{V}^2$/2	$\omega\cdot\mathbf{I}\omega$ /2
Eq. of Motion	d**P**/dt=**F**	d**L**/dt=**N**
Free Motion	Linear Motion	Free Rotation

4. Free Rotation

- = Torque-free (vs Keplerian Motion)
  - Separation of Ang. Vel. and Angles → Integrable
- Integrals: Energy, T; Ang. Mom., (L_X, L_Y, L_Z)
- Special Solution
  - Rotation Around one of Principal Axes
- General Solution
  - A=B: Trig. F., Free Polar Motion & Precession
  - A<B: Elliptic F. + Incomplete Elliptic Integrals

Special Solution

- AM Axis = one of Principal Axes
- Solution = Uniform Rotation
- Case of C-axis
  - Dynamics Part $L_A = L_B = 0, L_C = C\omega_0$
  - Kinematics Part $\psi = \psi_0, \theta = \theta_0, \phi = \phi_0 + \omega_0 t$
  - After Some Coord. Transf. $\psi = \theta = 0, \phi = \omega_0 t$
- Stability?

Stability of Special Sol.

- Assumption: Other Ang. Vel. Comp. are Small
- Linear Stability Theory: Growth Rate Eval.

$$L_A, L_B \ll L_C \sim L_0 \qquad L_A, L_B \propto e^{\lambda t}$$

$$\frac{\mathrm{d}L_A}{\mathrm{dt}} \sim -(b-c)L_0 L_B, \quad \frac{\mathrm{d}L_B}{\mathrm{dt}} \sim -(c-a)L_0 L_A$$

$$\to \lambda^2 = (b-c)(c-a)L_0^2 < 0$$

  - Cyclic Change of Indices (A,a → B,b → C,c)
- Stable: A- or C-axis ⇔ Unstable: B-axis

A=B<C

- Non-trivial Integrals
  - C-axis Component of **L**

$$\frac{\mathrm{d}L_C}{\mathrm{dt}} = 0 \to L_C = C\omega_0$$

- A- and B-Comp. of **L**: Harmonic Oscillation

$$\frac{\mathrm{d}L_A}{\mathrm{dt}} = -\Omega L_B$$

$$\frac{\mathrm{d}L_B}{\mathrm{dt}} = +\Omega L_A$$

$$\Omega = (a-c)L_C = \frac{C-A}{A}\omega_0$$

$$\left(\frac{2\pi}{\Omega}\right)_{\text{Earth}} \approx 303.438 \text{ day}$$

A=B<C (2)

- View in Body-Fixed CS: **L** = Slow Precession

$$L_A = L_{A0}\cos\Omega t - L_{B0}\sin\Omega t$$
$$L_B = L_{A0}\sin\Omega t + L_{B0}\cos\Omega t$$

- Similar Harm Osc. For Ang. Vel. Ω

= Free Polar Motion
= Euler Motion

$$\omega_A = \omega_{A0}\cos\Omega t - \omega_{B0}\sin\Omega t$$
$$\omega_B = \omega_{A0}\sin\Omega t + \omega_{B0}\cos\Omega t$$

- Earth's Euler Motion: P～303days

A=B<C (3)

$$L_A = G\sin\theta\sin\phi$$
$$L_B = G\sin\theta\cos\phi$$
$$L_C = G\cos\theta$$

$$G \equiv |\mathbf{L}|$$

- New Z-axis = AM Axis

$$\phi = \tan^{-1}\left(\frac{L_A}{L_B}\right) = \phi_0 - \Omega t,\ \ \theta = \theta_0 \equiv \cos^{-1}\left(\frac{L_C}{G}\right)$$

- Rot. Angle ϕ , Inclination θ -> Precession A. ψ

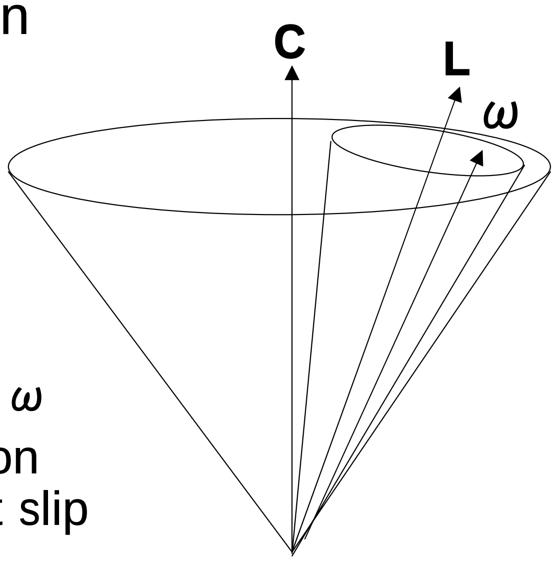
$$\frac{\mathrm{d}\psi}{\mathrm{dt}} = \frac{1}{\cos\theta}\left(\omega_0 - \frac{\mathrm{d}\phi}{\mathrm{dt}}\right) \rightarrow \psi = \psi_0 + aGt$$

- Solution: Slow Reverse Rotation + Constant Inclination + Fast Precession

Moving and Fixed Cones

- Poinsot's Expression

=Geometric View in Body-Fixed CS (A=B<C)
  - Fixed C.: C-axis
  - Moving C.: L-axis
  - Modulus: Rot. Axis ω
  - Moving Cone rolls on Fixed Cone without slip



A<B<C

- Eq. of Free Rotation (Principal CS + AM)

$$\frac{dL_A}{dt} = -(b-c)L_B L_C$$

$$\frac{dL_B}{dt} = -(c-a)L_C L_A$$

$$\frac{dL_C}{dt} = -(a-b)L_A L_B$$

- Two Conservatives

(Q: Confirm)

$$2T = aL_A^2 + bL_B^2 + cL_C^2$$

$$G^2 = L_A^2 + L_B^2 + L_C^2$$

A<B<C (2)

- Principal CS + Ang. Vel.
  - Parameters of Inertia

$$\alpha \equiv \frac{C-B}{A} > 0, \beta \equiv \frac{A-C}{B} < 0, \gamma \equiv \frac{B-A}{C} > 0$$

  - Identity on Param. Of Inertia (Q: Show)

$$\alpha + \beta + \gamma + \alpha\beta\gamma = 0$$

A<B<C (3)

- Eq. of Rotation (Principal CS + Ang. Vel.)

$$\frac{\mathrm{d}\omega_A}{\mathrm{dt}} = -\alpha\omega_B\omega_C$$

$$\frac{\mathrm{d}\omega_B}{\mathrm{dt}} = -\beta\omega_C\omega_A$$

$$\frac{\mathrm{d}\omega_C}{\mathrm{dt}} = -\gamma\omega_A\omega_B$$

- Two Conservatives

  Again (Q: Show)

$$2T = A\omega_A^2 + B\omega_B^2 + C\omega_C^2$$

$$G^2 = A^2\omega_A^2 + B^2\omega_B^2 + C^2\omega_C^2$$

A<B<C (4)

- Characteristics (Depend on Initial Cond.)
  - Charact. Mom. Inertia $D \equiv \frac{G^2}{2T}$ $A \le D \le C$
  - Inverse Charact. MoI $d \equiv \frac{1}{D}$ $a \ge d \ge c$

  = Inverse of Charact. MoI
  - Charact. Ang. Vel. $\omega_D \equiv \frac{2T}{G}$
  - Relation $D\omega_D = G$

A<B<C (5)

- Kirchhoff Transformation
- Using Characteristics
  - Time: t → u $u \equiv G\sqrt{(a-d)(b-c)}\,(t-t_0)$
  - Ang. Mom.: L_A, L_B, L_C → x, y, z

$$\frac{L_A}{G} \equiv \sqrt{\frac{d-c}{a-c}}\,x, \quad \frac{L_B}{G} \equiv \sqrt{\frac{d-c}{b-c}}\,y, \quad \frac{L_C}{G} \equiv \sqrt{\frac{a-d}{a-c}}\,z$$

  - Initial Condition $L_B = 0 \text{ when } t = t_0$

A<B<C (6)

- Transformed Eq. of Motion (Q: Confirm)
  - Q: Derive Kirchhoff Transf. s.t. Resulting This Form

$$\frac{\mathrm{d}x}{\mathrm{du}} = -yz, \quad \frac{\mathrm{d}y}{\mathrm{du}} = zx, \quad \frac{\mathrm{dz}}{\mathrm{du}} = -k^2 xy$$

- Modulus of Elliptic F.: k

$$k^2 \equiv \frac{(a-b)(d-c)}{(a-d)(b-c)} \geq 0$$

- Jacobi's Elliptic Func.

$$x = \mathrm{cn}(u;k),\ y = \mathrm{sn}(u;k),\ z = \mathrm{dn}(u;k)$$

A<B<C (7)

- Low Energy Assumption (b<d → k<1)
- New Z-axis
  = **L** axis

$$L_A = A\omega_A = G\sin\theta\sin\phi = G\sqrt{\frac{d-c}{a-c}}\mathrm{cn}(u;k)$$

$$L_B = B\omega_B = G\sin\theta\cos\phi = G\sqrt{\frac{d-c}{b-c}}\mathrm{sn}(u;k)$$

$$L_C = C\omega_C = G\cos\theta = G\sqrt{\frac{a-d}{a-c}}\mathrm{dn}(u;k)$$

- Polhode
  - AM Ellipse
  - AV Ellipse

$$(a-c)L_A^2 + (b-c)L_B^2 = (d-c)G^2$$

$$(C-A)A\omega_A^2 + (C-B)B\omega_B^2 = 2CT - G^2$$

A<B<C (8)

- Solution of Rotation Angle and Inclination
  - Wobbling and Slowly-decreasing ϕ (Q: Show)

$$\phi = \tan^{-1}\left(\frac{L_A}{L_B}\right) = \frac{\pi}{2} - \mathrm{am}(u;k) - \tan^{-1}\left(\frac{(1-\sigma)\,\mathrm{sn}(u;k)\,\mathrm{cn}(u;k)}{\sigma + (1-\sigma)\,\mathrm{sn}^2(u;k)}\right)$$

$$\sigma \equiv \sqrt{\frac{b-c}{a-c}} \leq 1$$

  - Oscillating Inclination θ (Q: Show)

$$\theta = \cos^{-1}\left(\frac{L_C}{G}\right) = \cos^{-1}\left(\sqrt{\frac{a-d}{a-c}}\,\mathrm{dn}(u;k)\right)$$

A<B<C (9)

- Precession Angle ψ : Incompl. El of 3rd Kind
  - Kinoshita (1992, CMDA)

$$\psi = \int \frac{aL_A \sin\phi + bL_B \cos\phi}{\sin\theta} \mathrm{dt}$$

$$= \psi_0 + cG(t - t_0) + \frac{a-c}{\sqrt{(a-d)(b-c)}} \mathrm{pn}(u; n, k)$$

- Parameter of Elliptic Integral

$$n \equiv \frac{a-b}{b-c} = \frac{1-\sigma^2}{\sigma^2}$$

A<B<C (10)

- Period of Rotation Angle

$$P_\phi \equiv \frac{4K}{\nu} = \frac{4K}{G\sqrt{(a-d)(b-c)}}$$

$$\beta \equiv \tan^{-1}\sqrt{\frac{d-c}{a-d}}$$

- Period of Precession Angle (Q: Show)

$$P_\psi = \frac{2\pi}{G\left[c+(a-c)\left\{\frac{k^2}{k^2+n}+\left(\frac{\pi}{2K}\right)\frac{\sqrt{n}\Lambda_0(\beta;k)}{\sqrt{(1+n)(k^2+n)}}\right\}\right]}$$

5. Elliptic Functions

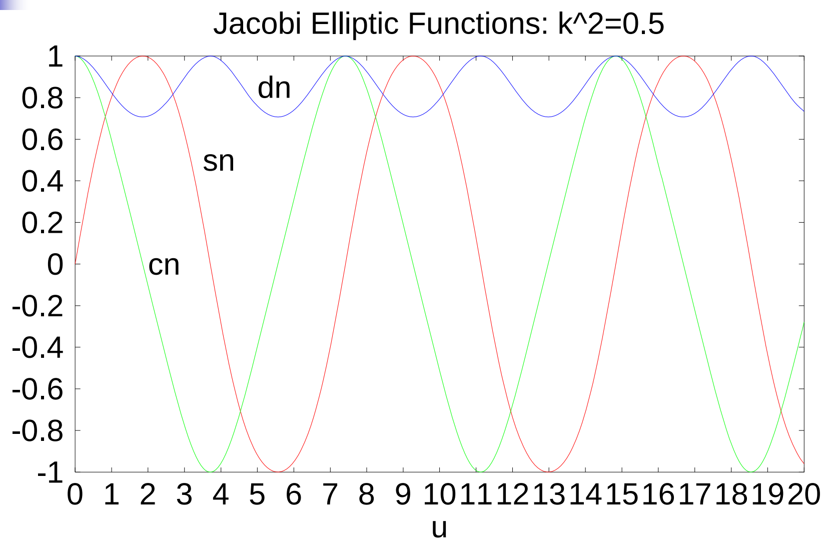
- Jacobi's Elliptic Functions (Jacobi 1829): sn, cn, dn
  - "Bible": Byrd and Friedman (1954)
- sn: Inverse Function of Incomplete Elliptic Integral of 1st Kind in Rational Form

$$u = \int_0^x \frac{\mathrm{d}s}{\sqrt{\left(1-s^2\right)\left(1-k^2s^2\right)}} \equiv \mathrm{sn}^{-1}x$$

$$\mathrm{cn}(u;k) \equiv \pm\sqrt{1-\mathrm{sn}^2(u;k)}, \mathrm{dn}(u;k) \equiv \sqrt{1-k^2\mathrm{sn}^2(u;k)}$$

- Argument: u, Modulus: k
  - Usually We Omit Modulus

Graphs of Elliptic F.



Amplitude Function

- Inverse Function of Incomplete Elliptic Integral of 1st Kind in Trigonometric Form

$$u=\int_0^{\varphi}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}\equiv \mathrm{am}^{-1}\varphi$$

- Amplitude Function $\varphi=\mathrm{am}(u;k)$
- Trigonometric Expr. of Elliptic Functions

$$\mathrm{sn}u=\sin\varphi,\ \mathrm{cn}u=\cos\varphi,\ \mathrm{dn}u=\sqrt{1-k^2\sin^2\varphi}$$

Modulus

- Modulus: k
- But, $m=k^2$ is more suitable
  - Clear from Definition of Incomplete EI
- Complimentary Modulus

$$k' = k_c \equiv \sqrt{1-k^2}$$

- Caution: Expression of k or m

$$\mathrm{sn}(u \mid m) \equiv \mathrm{sn}(u;k), \text{ etc.}$$

Formulas

- Two Identities

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1,\ \ \mathrm{dn}^2 u + k^2 \mathrm{sn}^2 u = 1$$

- Special Modulus
  - k=0 $\mathrm{sn}u, \mathrm{cn}u, \mathrm{dn}u \to \sin u, \cos u, 1$
  - k=1 $\mathrm{sn}u, \mathrm{cn}u, \mathrm{dn}u \to \tanh u, \mathrm{sech}u, \mathrm{sech}u$
- Special Argument: u=K/2
  - K(k) is Complete EI of 1st Kind

$$(\mathrm{sn}u, \mathrm{cn}u, \mathrm{dn}u) = \left(\frac{1}{\sqrt{1+k}}, \sqrt{\frac{k}{1+k}}, \sqrt{k} \right)$$

Notation

- Guderman Notation
  - Sine Amplitude=sn, Cosine Amplitude=cn
  - Delta Amplitude=dn, dn(u;k)= $\Delta(\phi)$
- Glaisher (fraction) notation
  - p,q,r = one of (s,c,d,n)
  - Examples

$$\mathrm{pq}(u;k) \equiv \frac{\mathrm{pr}(u;k)}{\mathrm{qr}(u;k)}$$

$$\mathrm{nd}(u;k) = \frac{1}{\mathrm{dn}(u;k)}, \mathrm{sd}(u;k) = \frac{\mathrm{sn}(u;k)}{\mathrm{dn}(u;k)}, \mathrm{cs}(u;k) = \frac{\mathrm{cn}(u;k)}{\mathrm{sn}(u;k)}$$

Inverse Function

- Base: 2-Argument Inverse Elliptic Function

$$x = r\,\mathrm{cn}(u;k),\ y = r\,\mathrm{sn}(u;k) \rightarrow u = \mathrm{atn2}(y, x; k)$$

- Using Incomplete EI of 1st Kind in Trig. Form

$$\mathrm{atn2}(y, x; k) = F(\mathrm{atan2}(y, x); k)$$

- General Expression of Amplitude Function

$$\mathrm{am}(u;k) = 2\pi \left[\frac{u}{4K(k)} \right]_{\mathrm{Gauss}} + \mathrm{atan2}(\mathrm{sn}(u;k), \mathrm{cn}(u;k))$$

Reciprocal Transf.

- Reduction of Modulus' Domain
  - Reciprocal & Imaginary Transf. → 0<k<1
  - Gauss & Landen Transf. → k~0 or k~1
- Reciprocal Transformation
  - k>1 to k<1

$$k \to k^{-1}$$

$$\mathrm{sn}(u;k) = k^{-1}\mathrm{sn}(ku;k^{-1})$$

$$\mathrm{cn}(u;k) = \mathrm{dn}(ku;k^{-1})$$

$$\mathrm{dn}(u;k) = \mathrm{cn}(ku;k^{-1})$$

Imaginary Transf.

- Imaginary Transformation
  - Pure Imaginary k to Real k

$$k \to \tilde{k} \equiv \sqrt{\frac{-k^2}{1-k^2}}$$

$$u \to \tilde{u} \equiv \sqrt{1-k^2}\,u$$

$$\mathrm{cn}(u;k) = \frac{\mathrm{cn}(\tilde{u};\tilde{k})}{\mathrm{dn}(\tilde{u};\tilde{k})}$$

$$\mathrm{sn}(u;k) = \frac{\sqrt{1-\tilde{k}^2}\,\mathrm{sn}(\tilde{u};\tilde{k})}{\mathrm{dn}(\tilde{u};\tilde{k})}$$

$$\mathrm{dn}(u;k) = \frac{1}{\mathrm{dn}(\tilde{u};\tilde{k})}$$

Landen's Transf.

- (Ascending) Landen's Transformation
  - Closer to 1

$$k \to \bar{k} \equiv \frac{2\sqrt{k}}{1+k}$$

$$u \to \bar{u} \equiv \frac{u}{1+\sqrt{1-\bar{k}^2}}$$

$$\operatorname{cn}(u;k) = \frac{\left(1+\sqrt{1-\bar{k}^2}\right)\operatorname{sn}(\bar{u};\bar{k})\operatorname{cn}(\bar{u};\bar{k})}{\operatorname{dn}(\bar{u};\bar{k})}$$

$$\operatorname{sn}(u;k) = \frac{1-\left(1+\sqrt{1-\bar{k}^2}\right)\operatorname{sn}^2(\bar{u};\bar{k})}{\operatorname{dn}(\bar{u};\bar{k})}$$

$$\operatorname{dn}(u;k) = \frac{1-\left(1-\sqrt{1-\bar{k}^2}\right)\operatorname{sn}^2(\bar{u};\bar{k})}{\operatorname{dn}(\bar{u};\bar{k})}$$

Gauss' Transformation

- = Descending Landen's Transf.
  - Closer to 0

$$k \to \hat{k} \equiv \left(\frac{k^2}{1+\sqrt{1-k^2}} \right)^2$$

$$u \to \hat{u} \equiv \frac{u}{1+\hat{k}}$$

$$\mathrm{cn}(u;k) = \frac{\mathrm{cn}(\hat{u};\hat{k})\,\mathrm{dn}(\hat{u};\hat{k})}{1+\hat{k}\,\mathrm{sn}^2(\hat{u};\hat{k})}$$

$$\mathrm{sn}(u;k) = \frac{(1+\hat{k})\,\mathrm{sn}(\hat{u};\hat{k})}{1+\hat{k}\,\mathrm{sn}^2(\hat{u};\hat{k})}$$

$$\mathrm{dn}(u;k) = \frac{1-\hat{k}\,\mathrm{sn}^2(\hat{u};\hat{k})}{1+\hat{k}\,\mathrm{sn}^2(\hat{u};\hat{k})}$$

Jacobi's Nome, q

- Useful in Fast Computation
- Def.: K, and its Compliment, K'

$$K'(k) \equiv K(k')$$

$$q \equiv \exp\left(\frac{-\pi K'}{K}\right) = \lambda\left(1 + 2\lambda^4 + 15\lambda^8 + 150\lambda^{12} + \cdots\right)$$

  - q is small when $k^2 < 1/2$
  - $k^2 > 1/2$ -> Dual Transf.

$$k \leq \frac{1}{\sqrt{2}} \rightarrow q \leq e^{-\pi} \sim 0.0432$$

- Expansion Factor

$$\lambda \equiv \frac{k^2}{2(1+k')\left(1+\sqrt{k'}\right)^2} \approx \frac{k^2}{16}$$

$$k' \equiv \sqrt{1-k^2}$$

Nome Expression

- Fraction Expr. Using Nome (Fast Computation)
  - After Reduction to Basic Domain: $0<k^2<1/2$

$$\mathrm{sn}(u;k)=\frac{\beta s y_s}{y_n},\mathrm{cn}(u;k)=\frac{\sqrt{k'}\beta c y_c}{y_n},\mathrm{dn}(u;k)=\frac{\sqrt{k'}y_d}{y_n}$$

$$s=\sin v, c=\cos v, v\equiv\frac{\pi u}{2K(k)}$$

$$\beta\equiv\frac{2\sqrt[4]{q}}{\sqrt{k}}=\frac{1+2\left(q+q^4+q^9+\cdots\right)}{1+q^2+q^6+q^{12}+\cdots}$$

  - Q: Draw Graphs of β & k as Function of q
- Normalized Elliptic Theta Functions: y(q)

q-Exp. of y-Functions

- Normalized Theta Function: Rapidly Converge

$$y_s = 1 - C_2 q^2 + C_4 q^6 - C_6 q^{12} + \cdots$$
$$y_c = 1 + S_3 q^2 + S_5 q^6 + S_7 q^{12} + \cdots$$
$$y_d = 1 + 2\left(T_2 q + T_4 q^4 + T_6 q^9 + \cdots\right)$$
$$y_n = 1 - 2\left(T_2 q - T_4 q^4 + T_6 q^9 + \cdots\right)$$

- Coefficients (= Chebychef Polynomials)
  - Recursively Computable $C_n \equiv \dfrac{\sin(n+1)\nu}{\sin\nu}, S_n \equiv \dfrac{T_n}{\cos\nu}, T_n \equiv \cos(n\nu)$

Fourier Expansion

- Not-so-rapid convergence as q-expansion

$$\mathrm{sn}(u;k) = \frac{2\pi}{kK(k)} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1-q^{2n+1}} \sin(2n+1)v$$

$$\mathrm{cn}(u;k) = \frac{2\pi}{kK(k)} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \cos(2n+1)v$$

$$\mathrm{dn}(u;k) = \frac{\pi}{2K(k)} + \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \frac{q^{n}}{1+q^{2n}} \cos 2nv$$

$$\mathrm{am}(u;k) = v + \sum_{n=1}^{\infty} \frac{2q^{n}}{n(1+q^{2n})} \sin 2nv$$

$$v \equiv \frac{\pi u}{2K(k)}$$

Numerical Computation

- Gauss' Arithmetic-Geometric-Mean (AGM)
  - Common Conv. of Two Var. $a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n}$
- Landen's Transf. of Ampl. F.

$$u_{n+1} = \frac{u_n + v_n}{2}, v_{n+1} = \sqrt{u_n v_n}, \varphi_{n+1} = \frac{\varphi_n + \sin^{-1}\left(k_n \sin \varphi_n\right)}{2} \qquad v_n \equiv k_n u_n$$

- Algorithm (Q: Write a Computer Code)
  - Start ($u_0=u, v_0=ku$) and Repeat Transf. while Save $k_j=v_j/u_j$
  - After Convergence: $u_n=v_n$, i.e. $k_n=1$, then $\phi_n=u_n$
  - Start (ϕ_n) and Repeat Reverse Transf. up to ϕ_0 using k_j
  - Details: Numerical Recipe, 2nd Ed.

Incomplete Elliptic Integrals

- Jacobi (1850): Indefinite Integral of quartic rational
- 1st Kind

$$F(x;k)=\int_0^x \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

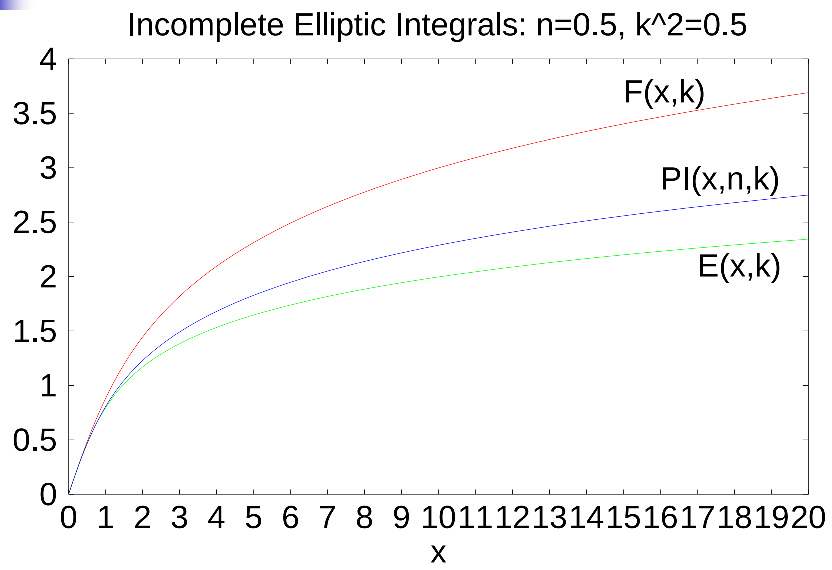
- 2nd Kind

$$E(x;k)=\int_0^x \sqrt{\frac{1-s^2}{1-k^2s^2}}\,ds$$

- 3rd Kind: Parameter, n = $-\alpha^2$

$$\Pi(x;n,k)\equiv\int_0^x \frac{ds}{(1+ns^2)\sqrt{(1-s^2)(1-k^2s^2)}}$$

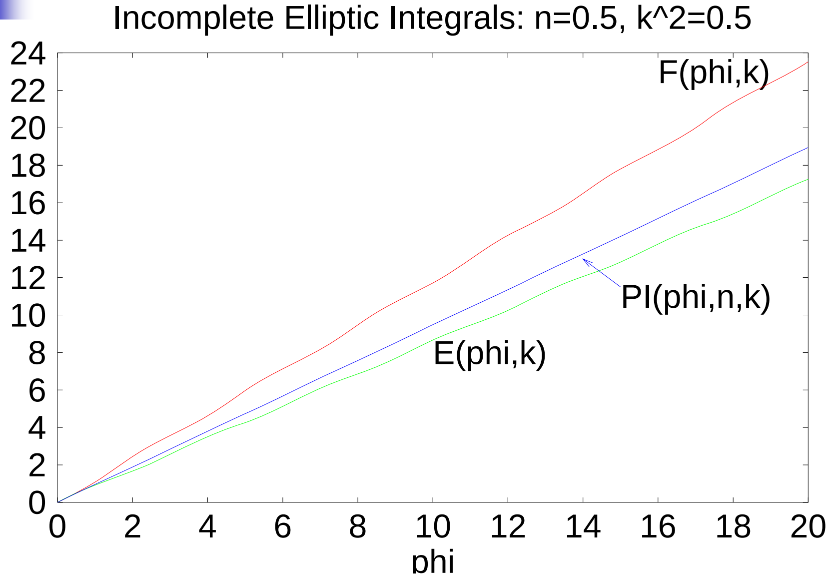
Incomplete EI (2)



Incomplete EI (3)

- Trigonometric Representation (Legendre)
  - Most Popular, Note: Difference in Argument
- 1st Kind $$F(\varphi;k)=\int_0^{\varphi}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}$$
- 2nd Kind $$E(\varphi;k)=\int_0^{\varphi}\frac{\cos^2\theta\,\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}$$
- 3rd Kind $$\Pi(\varphi;n,k)=\int_0^{\varphi}\frac{\mathrm{d}\theta}{\left(1+n\sin^2\theta\right)\sqrt{1-k^2\sin^2\theta}}$$

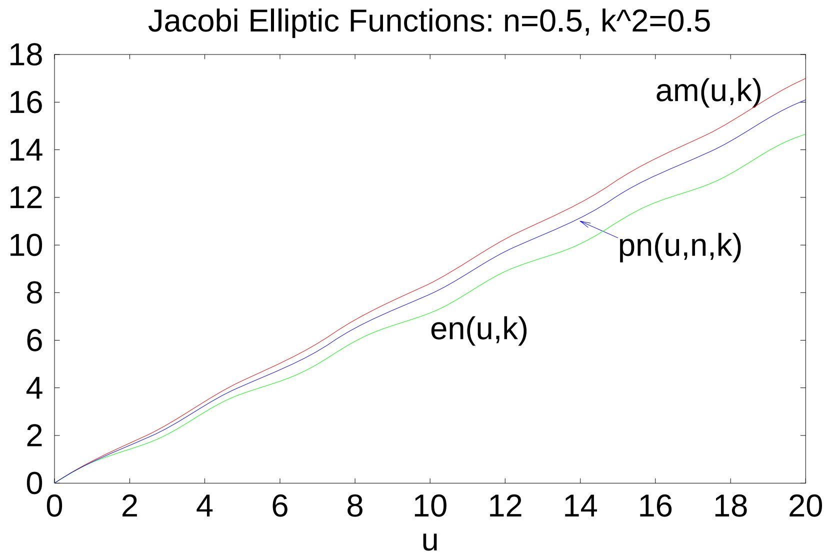
Incomplete EI (4)



Incomplete EI (5)

- Elliptic Function Expression (Jacobi)
  - Note: Difference in Argument
  - Introducing New Symbols: en, pn
- 1st Kind $F(u;k) = u$
- 2nd Kind $E(u;k) = \int_0^u \mathrm{cn}^2(v;k)\,\mathrm{d}v = \mathrm{en}(u;k)$
- 3rd Kind $\Pi(u;n,k) \equiv \int_0^u \frac{\mathrm{d}v}{1+n\,\mathrm{sn}^2(v;k)} = \mathrm{pn}(u;n,k)$

Incomplete EI (6)



Incomplete EI (7)

- Another Expr. Of Incomplete EI of 2nd Kind

$$E(u;k) = \left(\frac{K(k)}{E(k)}\right) u + Z(u;k)$$

- Jacobi's Zeta F. = Periodic Part of E(u;k)

$$Z(u;k) \equiv \mathrm{zn}(u;k)$$
$$\simeq \frac{\pi}{K(k)}\left(\frac{2q\sin 2v - 4q^4 \sin 4v + 9q^9 \sin 6v}{1 - 2q\cos 2v + 2q^4 \cos 4v - 2q^9 \cos 6v}\right)$$

$$v \equiv \frac{\pi u}{2K(k)}$$

Incomplete EI (8)

- Special Case
  - k=0

$$E(\varphi;0)=F(\varphi;0)=\varphi$$

$$\Pi(\varphi;n,0)=\frac{\tan^{-1}\left(\sqrt{1+n}\tan\varphi\right)}{\sqrt{1+n}}$$

  - k=1

$$E(\varphi;1)=\sin\varphi$$

$$F(\varphi;1)=\ln\left(\frac{1+\sin\varphi}{\cos\varphi}\right)$$

$$\Pi(\varphi;n,1)=\frac{1}{n+1}\left[n\tan^{-1}(n\sin\varphi)+\frac{1}{2}\log\left|\frac{1-\sin\varphi}{1+\sin\varphi}\right|\right]$$

  - n=-k^2

$$\Pi(\varphi;-k^2,k)=\frac{1}{1-k^2}\left[E(\varphi;k)-\frac{k^2\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right]$$

Partial by Argument

- Partial Derivative w.r.t. Argument, u (Q: Show)
- Fixing Modulus, k, and Parameter, n

$$\frac{\partial \operatorname{sn}(u;k)}{\partial \mathrm{u}} = \operatorname{cn}u\operatorname{dn}u$$

$$\frac{\partial \operatorname{cn}(u;k)}{\partial \mathrm{u}} = -\operatorname{sn}u\operatorname{dn}u$$

$$\frac{\partial \operatorname{dn}(u;k)}{\partial \mathrm{u}} = -k^2\operatorname{sn}u\operatorname{cn}u$$

$$\frac{\partial \operatorname{am}(u;k)}{\partial \mathrm{u}} = \operatorname{dn}u$$

$$\frac{\partial \operatorname{en}(u;k)}{\partial \mathrm{u}} = \operatorname{cn}^2u$$

$$\frac{\partial \operatorname{pn}(u;n,k)}{\partial \mathrm{u}} = \frac{1}{1+n\operatorname{sn}^2(u;k)}$$

Partial by Modulus

- Partial Derivative w.r.t. m=k^2 (Q: Show)

$$\frac{\partial \mathrm{sn}(u|m)}{\partial \mathrm{m}} = \mathrm{cn}(u|m)\frac{\partial \mathrm{am}(u|m)}{\partial \mathrm{m}}$$

$$\frac{\partial \mathrm{cn}(u|m)}{\partial \mathrm{m}} = -\mathrm{sn}(u|m)\frac{\partial \mathrm{am}(u|m)}{\partial \mathrm{m}}$$

$$\frac{\partial \mathrm{dn}(u|m)}{\partial \mathrm{m}} = \frac{-\mathrm{sn}(u|m)}{2\mathrm{dn}(u|m)}\left[\mathrm{sn}(u|m)+2\mathrm{cn}(u|m)\frac{\partial \mathrm{am}(u|m)}{\partial \mathrm{m}}\right]$$

$$\frac{\partial \mathrm{am}(u|m)}{\partial \mathrm{m}} = \mathrm{dn}(u|m)\left[\frac{\mathrm{pn}(u;-m|m)-u}{2m}\right]$$

Partial by Parameter

- Partial Derivatives w.r.t. m or n (Q: Show)

$$\frac{\partial \mathrm{en}(u\,|\,m)}{\partial \mathrm{m}} = \frac{\mathrm{en}(u\,|\,m) - u}{2m}$$

$$\frac{\partial \mathrm{pn}(u;n\,|\,m)}{\partial \mathrm{m}} = \frac{1}{2}\left[u + \frac{\mathrm{pn}(u;-m\,|\,m) - \mathrm{pn}(u;n\,|\,m)}{m+n}\right]$$

$$\frac{\partial \mathrm{pn}(u;n\,|\,m)}{\partial \mathrm{n}} = \frac{1}{2(1+n)}\left[\frac{\mathrm{pn}(u;n\,|\,m) - u}{n} + \frac{1-m}{m+n}\mathrm{pn}(u;-m\,|\,m) + \frac{\mathrm{sn}(u\,|\,m)\,\mathrm{cn}(u\,|\,m)}{\left\{1 + n\,\mathrm{sn}^2(u\,|\,m)\right\}\mathrm{dn}(u\,|\,m)}\right]$$

Numerical Evaluation of Incomplete EI

- Numer. Evaluation of General Incomplete EI (Fukushima and Ishizaki 1994b, CMDA)

$$G(\varphi; n_c, m_c, a, b) \equiv \int_0^{\varphi} \frac{a\cos^2\theta + b\sin^2\theta}{\left(\cos^2\theta + n_c\sin^2\theta\right)\sqrt{\cos^2\theta + m_c\sin^2\theta}}\mathrm{d}\theta$$

- Basic Expression Formulas (Q: Confirm)

$$\lambda F(\varphi;k) + \mu E(\varphi;k) = G\left(\varphi; 1, 1-k^2, \lambda+\mu, \lambda+\mu\left(1-k^2\right)\right)$$

$$\lambda F(\varphi;k) + \mu\Pi(\varphi;n,k) = G\left(\varphi; 1+n, 1-k^2, \lambda+\mu, \lambda(1+n)+\mu\right)$$

General Elliptic F.

- Elliptic F. based on General Incomplete EI

$$\mathrm{gn}(u;n\,|\,m,a,b)\equiv G\left(\mathrm{am}(u\,|\,m);1+n,1-m,a,b\right)$$

- Basic Expression Formulas

$$\lambda u+\mu\,\mathrm{en}(u\,|\,m)=\mathrm{gn}\left(u;1,1-m,\lambda+\mu,\lambda+\mu(1-m)\right)$$

$$\lambda u+\mu\,\mathrm{pn}(u;n\,|\,m)=\mathrm{gn}\left(u;1+n,1-m,\lambda+\mu,\lambda(1+n)+\mu\right)$$

- Application No.1: (Q: Confirm)

$$\mathrm{zn}(u\,|\,m)=\mathrm{gn}\left(u;1,1-m,1-\frac{K}{E},1-m-\frac{K}{E}\right)$$

Application of General EF

- Application No.2 (Q: Confirm)
  - Needed in Partial Computations

$$\frac{(1-m)u-\mathrm{en}(u\,|\,m)}{m}=\mathrm{gn}(u;1,1-m,1,0)$$

$$\frac{u-\mathrm{pn}(u;n\,|\,m)}{n}=\mathrm{gn}(u;1+n,1-m,0,1)$$

$$\mathrm{fn}(u\,|\,m)\equiv\frac{\mathrm{pn}(u;-m\,|\,m)-u}{m}=\mathrm{gn}(u;1-m,1-m,0,1)$$

Complete EI

- Complete EI = Definite Integral
- 1st Kind $$K(k) \equiv F(1;k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$
- 2nd Kind $$E(k) \equiv E(1;k) = \int_0^1 \sqrt{\frac{1-x^2}{1-k^2x^2}}dx$$
- 3rd Kind

$$\Pi(n,k) \equiv \Pi(1;n,k) = \int_0^1 \frac{ds}{(1+ns^2)\sqrt{(1-s^2)(1-k^2s^2)}}$$

Complete EI (2)

- Another Expr. Of Complete EI of 3rd Kind (n>0)

$$\Pi(n,k)=\frac{k^2K(k)}{k^2+n}+\left(\frac{\pi}{2}\right)\frac{\sqrt{n}\Lambda_0(\beta;k)}{\sqrt{(1+n)(k^2+n)}}$$

$$\beta\equiv\tan^{-1}\frac{\sqrt{n}}{k}$$

- Heuman's Lambda Function

$$\Lambda_0(\beta;k)\equiv\frac{2}{\pi}\left[\left\{E(k)-K(k)\right\}F(\beta;k')+K(k)E(\beta;k')\right]$$

Complete EI (3)

- (Real) Period of Elliptic Functions: 4K
  - K(0)= π /2, K(1)=∞
- Complimentary Relations

$$K'(k) \equiv K(k'), E'(k) \equiv E(k') \qquad k' \equiv \sqrt{1-k^2}$$

- Omitting Modulus k Frequently: K,K',E,E'
- Legendre's Relation

$$\Lambda_0\left(\frac{\pi}{2}, k\right) = 1 \rightarrow EK' + E'K - KK' = \frac{\pi}{2}$$

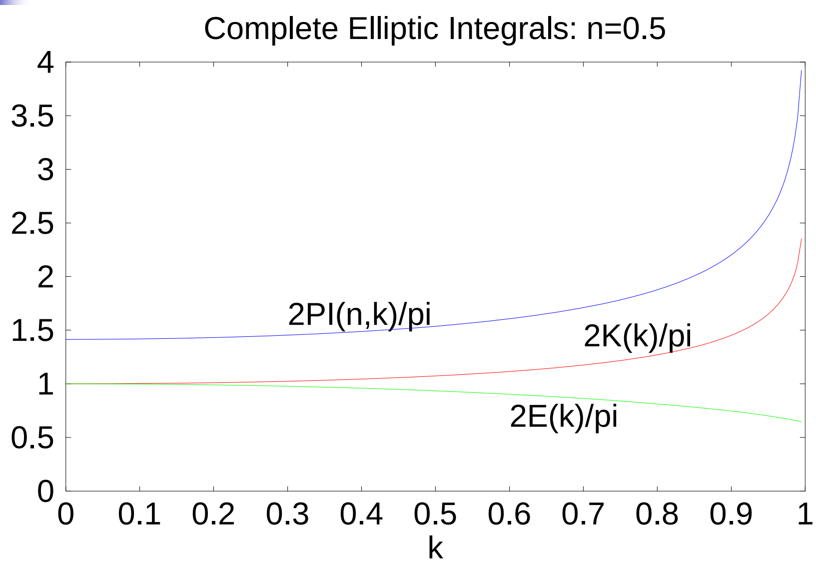
Dual Transformation

- = Complimentary Transf. = Prime Transf.

  $k \to k' \equiv \sqrt{1-k^2}$ $m \to m' \equiv 1-m$

  - $0<k^2<1/2 \to 1/2<k^2<1$
- Complete Elliptic Integrals $K,E \to K',E'$
- Nome $q \to q' \equiv \exp\left(\dfrac{\pi^2}{\log q}\right)$

Complete EI (2)



Numerical Evaluation of Complete EI

- Inefficient: Calling Routines of Incomplete EI
- Standard: Using AGM (Numerical Recipe, etc)
- Practical: Modulus Transf. + q-Expansion
- Ex: $k^2<1/2$
  - 1st Kind $$K(k) = 2\pi\left(\frac{1+2q^4+\cdots}{1+\sqrt{k'}}\right)^2$$
  - 2nd Kind $$E(k) = K(k) - \left(\frac{2\pi^2}{K(k)}\right)\left(\frac{q-4q^4+9q^9-\cdots}{1-2\left(q-q^4+q^9-\cdots\right)}\right)$$

Small Modulus

- Earth/Moon: m~0 $k_{\mathrm{Earth}} \sim 3\times10^{-8}, k_{\mathrm{Moon}} \sim 5\times10^{-6}$
- Enough for 1st Order Exp. w.r.t. m

$$q \simeq \frac{m}{16} \quad k' \simeq 1-\frac{m}{2} \quad K \simeq \frac{\pi}{2}\left(1+\frac{m}{2}\right) \quad E \simeq \frac{\pi}{2} \quad v \simeq \left(1-\frac{m}{2}\right)u$$

$$\mathrm{sn}u = \left(1+\frac{m}{4}\cos^2 u\right)\sin v \quad \mathrm{cn}u = \left(1-\frac{m}{4}\sin^2 u\right)\cos v$$

$$\mathrm{dn}u = 1-\frac{m}{2}\sin^2 u \quad \mathrm{zn}u \simeq \frac{m}{4}\sin 2u \quad \mathrm{en}u = \left(1+\frac{m}{2}\right)u + \mathrm{zn}u$$

6. Forced Rotation

- Under Torque
  - Ex: Earth Precession/Nutation, Moon's Rotation
- Strength of Perturbation
  - Comparison of Rot. Energy vs Torque
- Unperturbed State
  - Simplification: $A=B<C$, $C-A \ll C$
  - Uniform Rotation around C-axis
- Type of Perturbation
  - Figure Axis Motion, Variation of Rot. Speed

Torque Expression

- In Inertial CS: Complicated (Q: Confirm)

$$N_X = \frac{3\mu}{r^5}\left[r_X r_Y I_{ZX} - r_Z r_X I_{XY} + \left(r_Y^2 - r_Z^2\right) I_{YZ} + r_Y r_Z \left(I_{ZZ} - I_{YY}\right)\right]$$

  - Cyclic Change of Indices（X→Y→Z）

- Moment-of-Inertia Components

$$I_{XY} = A\left(\mathbf{e}_A\right)_X \left(\mathbf{e}_A\right)_Y + B\left(\mathbf{e}_B\right)_X \left(\mathbf{e}_B\right)_Y + C\left(\mathbf{e}_C\right)_X \left(\mathbf{e}_C\right)_Y$$

$$\mathbf{I} = A\mathbf{e}_A \otimes \mathbf{e}_A + B\mathbf{e}_B \otimes \mathbf{e}_B + C\mathbf{e}_C \otimes \mathbf{e}_C$$

Torque Expression (2)

- In Principal CS: Plain

$$\begin{pmatrix} N_A \\ N_B \\ N_C \end{pmatrix} = \left(\frac{3\mu}{r^5}\right) \begin{pmatrix} (C-B)\,r_B r_C \\ (A-C)\,r_C r_A \\ (B-A)\,r_A r_B \end{pmatrix}$$

$N_A = \mathbf{N} \bullet \mathbf{e}_A, \ldots$ $r_A = \mathbf{r} \bullet \mathbf{e}_A, \ldots$

Torque Magnitude

- Dim. Of Torque = Energy
- Ratio of Rotational Energy and Torque
  - Earth: Tiny

$$\frac{N_E}{T_E} \approx \frac{3(C_E - A_E)}{C_E \omega_E^2}\left(\frac{\mu_S}{r_{SE}^3} + \frac{\mu_M}{r_{ME}^3}\right)\sin 2\varepsilon_E \simeq 1.680\times 10^{-7}$$

  - Moon: Small

$$\varepsilon_M \simeq 1°32'33''$$

$$\frac{N_M}{T_M} \approx \frac{3(C_M - A_M)}{C_M \omega_M^2}\left(\frac{\mu_E}{r_{EM}^3}\right)\sin 2\varepsilon_M \simeq 9.91\times 10^{-5}$$

A=B<C

- N_C=0 → Integral Exists
  - C-comp. of Ang. Mom.

$$\frac{\mathrm{d}L_C}{\mathrm{dt}} = 0 \rightarrow L_C = G_0$$

- Torque

$$\mathbf{N} = \frac{3\mu\left(C - A\right) r_C}{r^5} \mathbf{r} \times \mathbf{e}_C$$

- Poisson Approximation

$$L_A = L_B = 0 \rightarrow \mathbf{L} = L_C \mathbf{e}_C$$

- Eq. of Motion of ψ, θ

$$\frac{\mathrm{d}\mathbf{e}_C}{\mathrm{dt}} = \frac{\mathbf{N}}{G_0}$$

A=B<C (2)

- Eq. of Rotation in Inertial CS

$$\mathbf{e}_C = \begin{pmatrix} \sin\theta\sin\psi \\ -\sin\theta\cos\psi \\ \cos\theta \end{pmatrix} \qquad G_0 \frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} \sin\theta\sin\psi \\ -\sin\theta\cos\psi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} N_X \\ N_Y \\ N_Z \end{pmatrix}$$

- Eq. of Euler Angles in Poisson Approx.
  - Z-comp. of Torque

    Unnecessary (Why?)

$$\frac{\mathrm{d}\phi}{\mathrm{dt}} = \omega_C - \left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)\cos\theta$$

$$\frac{\mathrm{d}\psi}{\mathrm{dt}} = \frac{N_X\cos\psi + N_Y\sin\psi}{G_0\sin\theta}$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}} = \frac{N_X\sin\psi - N_Y\cos\psi}{G_0\cos\theta}$$

A~B<C

- Laplacian Approximation
  - Ignore (A-B)/C in case of A<B<C

$$\mathbf{N} \simeq \frac{3\mu\left(2C - A - B\right) r_C}{2r^5} \mathbf{r} \times \mathbf{e}_C$$

- Dyn. Flattening, H + Poisson Approx.

$$\frac{C - A}{C} \rightarrow H \equiv \frac{2C - A - B}{2C}$$

Planar Circular Motion

- Pert. Body = Circular Orbit on XY Plane
  - Ex: Sun's Pert. On Earth Rotation
  - L: Helioc. Long. of Earth $r_C = \mathbf{r}\cdot\mathbf{e}_C = \bar{r}\sin\theta\sin(L-\psi)$

$$\mathbf{r} = -\bar{r}\begin{pmatrix}\cos L \\ \sin L \\ 0\end{pmatrix} \qquad \mathbf{r}\times\mathbf{e}_C = \bar{r}\begin{pmatrix}-\cos\theta\sin L \\ \cos\theta\cos L \\ \sin\theta\cos(L-\psi)\end{pmatrix}$$

- Torque in Inertial CS (Circular Approx.)

$$\begin{pmatrix}N_X \\ N_Y\end{pmatrix} = \left[\frac{3\mu(C-A)}{2\bar{r}^3}\right]\sin 2\theta\sin(L-\psi)\begin{pmatrix}-\sin L \\ \cos L\end{pmatrix}$$

Precession Solution

- Eq. of Prec. Angle + Inclination (Circular)

$$\frac{d\psi}{dt} = -\kappa\cos\theta\left[1-\cos 2\left(L-\psi\right)\right] \qquad \kappa = \frac{3\mu\left(C-A\right)}{2C\omega_{C0}\bar{r}^3}$$

$$\frac{d\theta}{dt} = -\kappa\sin\theta\sin 2\left(L-\psi\right) \qquad \psi^{(1)}\left(t\right) = \psi_0 - pt$$

- 1st Approx.: Precession $\qquad \theta^{(1)}\left(t\right) = \theta_0$
- Precession Const. $p = \kappa\cos\theta_0$
- Initial Cond. (Earth) $\psi_0 = 0, \theta_0 = -\varepsilon_0 < 0$

Luni-solar Precession

- Earth $\omega_{C0} \approx 6.30039\ \mathrm{rad/day}$ $\quad \dfrac{C_E - A_E}{C_E} \approx 3.28474 \times 10^{-3}$

  $\cos \varepsilon_0 \approx 0.917482$
- Moon $\dfrac{\mu_M}{\bar{r}_{EM}^{\ 3}} \approx \left(\dfrac{M_M}{M_E + M_M} \right) n_\ell^2 \approx 6.31786 \times 10^{-4} \left(\mathrm{rad/day}\right)^2$
- Sun $\dfrac{\mu_S}{\bar{r}_{SE}^{\ 3}} \approx n_{\ell'}^2 \approx 2.95908 \times 10^{-4} \left(\mathrm{rad/day}\right)^2$
- Luni-solar Prec.: Crude Estimate $\kappa \approx 37.2229''/\mathrm{jy}$

  $p \approx \left(\kappa + \kappa'\right) \cos \varepsilon_0 \approx 50.1467''/\mathrm{jy}$ $\quad \kappa' \approx 17.4340''/\mathrm{jy}$

Short Period Nutation

- 2nd Approx. = Main Short Period Nutation

$$\psi^{(2)}(t) = \psi_0 - pt + \delta_1^{(2)}\psi \qquad \theta^{(2)}(t) = \theta_0 + \delta_1^{(2)}\theta$$

$$\delta_1^{(2)}\psi = N_{2L}\cos\theta_0 \sin 2\left(L - \psi^{(1)}\right)$$

$$\delta_1^{(2)}\theta = N_{2L}\sin\theta_0 \cos 2\left(L - \psi^{(1)}\right)$$

$$N_{2L} = \frac{\kappa}{2(n_L + p)} \qquad n_L = \frac{\mathrm{d}L}{\mathrm{dt}}$$

Elliptic Orbit

- Pert. Body: Elliptic Orbit in XY Plane
  - Mean Orb. Long. L Mean Anomaly ℓ
  - Pericenter Long. $\varpi \equiv L - \ell$
  - Semi-major Axis a Eccentricity e
- Ignoring 2nd and Higher Order of e

$$r_X = a\left(\cos L + \frac{e}{2}\cos\left(L+\ell\right) - \frac{3e}{2}\cos\varpi\right)$$

$$r_Y = a\left(\sin L + \frac{e}{2}\sin\left(L+\ell\right) - \frac{3e}{2}\sin\varpi\right)$$

$$r_Z = 0$$

$$r = a\left(1 - e\cos\ell\right)$$

Elliptic Orbit (2)

$$r_C = \mathbf{r}\cdot\mathbf{e}_C = -a\sin\theta\left[\sin(L-\psi)+\frac{e}{2}\sin(L-\psi+\ell)-\frac{3e}{2}\sin(\varpi-\psi)\right]$$

$$\mathbf{r}\times\mathbf{e}_C = a\begin{pmatrix} \cos\theta\left[\sin L+\frac{e}{2}\sin(L+\ell)-\frac{3e}{2}\sin\varpi\right] \\ -\cos\theta\left[\cos L+\frac{e}{2}\cos(L+\ell)-\frac{3e}{2}\cos\varpi\right] \\ -\sin\theta\left[\cos(L-\psi)+\frac{e}{2}\cos(L-\psi+\ell)-\frac{3e}{2}\cos(\varpi-\psi)\right] \end{pmatrix}$$

Elliptic Orbit (3)

- Torque in Inertial CS (Elliptic Approx.)

$$\begin{pmatrix} N_X \\ N_Y \end{pmatrix} = -N_0 \begin{pmatrix} \sin L + \frac{e}{2}\sin(L+\ell) - \frac{3e}{2}\sin\varpi \\ -\cos L - \frac{e}{2}\cos(L+\ell) + \frac{3e}{2}\cos\varpi \end{pmatrix}$$

$$N_0 \equiv -\left[\frac{3\mu(C-A)}{2a^3}\right]\sin 2\theta(1+5e\cos\ell) \times\left[\sin(L-\psi)+\frac{e}{2}\sin(L-\psi+\ell)-\frac{3e}{2}\sin(\varpi-\psi)\right]$$

Elliptic Orbit (4)

- Eq. of Prec. Angle and Inclination (Elliptic)

$$\frac{\mathrm{d}\psi}{\mathrm{dt}} = -\kappa\cos\theta\Big[1-\cos 2\left(L-\psi\right)+3e\cos\ell - \frac{7e}{2}\cos\left(2L-2\psi+\ell\right)+\frac{e}{2}\cos\left(2L-2\psi-\ell\right)\Big]$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}} = -\kappa\sin\theta\Big[\sin 2\left(L-\psi\right)+\frac{7e}{2}\sin\left(2L-2\psi+\ell\right) - \frac{e}{2}\sin\left(2L-2\psi-\ell\right)\Big]$$

Elliptic Terms

- Additional 2nd Approx. Sol. (Elliptic Terms)
  - Moon: 1 month, 10 days, 15 days

$$\delta_2^{(2)}\psi = -\cos\theta_0\left[N_\ell \sin\ell - N_{2L+\ell}\sin\left(2L-2\psi^{(1)}+\ell\right) + N_{2L-\ell}\sin\left(2L-2\psi^{(1)}-\ell\right)\right]$$

$$\delta_2^{(2)}\theta = \sin\theta_0\left[N_{2L+\ell}\cos\left(2L-2\psi^{(1)}+\ell\right) - N_{2L-\ell}\cos\left(2L-2\psi^{(1)}-\ell\right)\right]$$

$$N_\ell = \frac{3\kappa e}{n_\ell}, N_{2L+\ell} = \frac{7\kappa e}{2\left(2n_L+2p+n_\ell\right)}, N_{2L-\ell} = \frac{\kappa e}{2\left(2n_L+2p-n_\ell\right)}$$

$$N_{2L} : N_\ell \approx 1:6e \qquad N_\ell : N_{2L+\ell} : N_{2L-\ell} \approx 18:7:3 \qquad n_\ell = \frac{\mathrm{d}\ell}{\mathrm{d}t} \approx n_L$$

Rotation Angle

- Eq. of Rot. Angle

$$\frac{\mathrm{d}\phi}{\mathrm{dt}} = \omega_C - \left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)\cos\theta$$

- 1st Approx.: Unif. Rot.

$$\phi^{(1)}(t) = \phi_0 + \omega_0 t$$

$$\omega_0 = \omega_{C0} + p\cos\theta_0$$

- 2nd Approx.: Variation due to Nutation
  - = Difference between GAST and GMST

$$\phi^{(2)}(t) = \phi_0 + \omega_0 t + \Delta\phi$$

$$\delta\phi = -(\Delta\psi)\cos\theta_0$$

Introduction of UT1

- UT1 = Modified Rotation Angle of Earth

$$\mathrm{UT1} \equiv \int \omega_C \mathrm{dt} = \phi - \int \cos\theta \left(\frac{\mathrm{d}\psi}{\mathrm{dt}} \right) \mathrm{dt}$$

- Assumptions: 1) Rigid Earth, 2) A=B,
  3) Torque induced by External Mass points
- Important Conclusion $\mathrm{UT1} = \omega_C t$
  - UT1 is A Realization of Newton's Absolute Time Independently on Perturbations

Departure point

- Newcomb's Departure Point
  - = Guinot's Non Rotating Origin (NRO)
- Definition
  - Point D on Equator (= AB Plane) of Principal CS such that UT1 = Arc Length of DA

$$\mathrm{UT1} = \overset{\frown}{\mathrm{DA}} = \overset{\frown}{\mathrm{DN}} + \overset{\frown}{\mathrm{NA}}$$

$$\overset{\frown}{\mathrm{NA}} = \phi, \quad \overset{\frown}{\mathrm{DN}} = \int \cos\theta \, \mathrm{d}\psi$$

Inclined Orbit

- Pert. Body in Inertial CS
  - Origin=Rotating Body's Barycenter
- Coordinates Expression

$$\mathbf{r} = \begin{pmatrix} r_X \\ r_Y \\ r_Z \end{pmatrix}$$

$$r_C = \mathbf{r} \cdot \mathbf{e}_C = r_X \sin\theta \sin\psi - r_Y \sin\theta \cos\psi + r_Z \cos\theta$$

$$\mathbf{r} \times \mathbf{e}_C = \begin{pmatrix} r_Y \cos\theta + r_Z \sin\theta \cos\psi \\ -r_X \cos\theta + r_Z \sin\theta \sin\psi \\ -r_X \sin\theta \cos\psi - r_Y \sin\theta \sin\psi \end{pmatrix}$$

Inclined Orbit (2)

- Torque in Inertial CS (Inclined Orbit)

$$N_X = \frac{-3\mu(C-A)}{r^5}\Big[r_X r_Y \sin\theta\cos\theta\sin\psi + r_X r_Z \sin^2\theta\sin\psi\cos\psi$$
$$+r_Y r_Z\left(\cos^2\theta - \sin^2\theta\cos^2\psi\right) + \left(r_Z^2 - r_Y^2\right)\sin\theta\cos\theta\cos\psi\Big]$$

$$N_Y = \frac{-3\mu(C-A)}{r^5}\Big[r_X r_Y \sin\theta\cos\theta\cos\psi - r_Y r_Z \sin^2\theta\sin\psi\cos\psi$$
$$-r_X r_Z\left(\cos^2\theta - \sin^2\theta\cos^2\psi\right) + \left(r_Z^2 - r_X^2\right)\sin\theta\cos\theta\sin\psi\Big]$$

Inclined Orbit (3)

- Eq. of Precession/Inclination (Poisson Approx.)
  - Q: Show

$$\frac{\mathrm{d}\psi}{\mathrm{dt}}=\frac{3\mu(C-A)}{C\omega_C r^5}\left[r_X r_Y\cos\theta\sin 2\psi+r_Y r_Z\left(\frac{\cos 2\theta}{\sin\theta}\right)\cos\psi\right.$$
$$\left.-r_X r_Z\left(\frac{\cos 2\theta}{\sin\theta}\right)\sin\psi-r_X^2\cos\theta\sin^2\psi-r_Y^2\cos\theta\cos^2\psi+r_Z^2\cos\theta\right]$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}}=\frac{3\mu(C-A)}{C\omega_C r^5}\left[-r_X r_Y\sin\theta\cos 2\psi-r_Y r_Z\cos\theta\sin\psi\right.$$
$$\left.+r_X r_Z\cos\theta\cos\psi+\left(r_X^2-r_Y^2\right)\sin\theta\sin\psi\cos\psi\right]$$

Inclined Circular Orbit

- Inclined but Circular Orbit $r = \bar{r}$
  - Ex.: Moon in Earth Rotation
  - Constant Orbital Inclination $I = I_0$
  - Slowly Moving Ascending Node $\Omega = \Omega_0 + n_\Omega t$
  - Argument of Latitude $F \equiv L - \Omega$
- Pert. Body's Position in Inertial CS (Q: Show)

$$\mathbf{r} = \begin{pmatrix} r_X \\ r_Y \\ r_Z \end{pmatrix} = \bar{r} \begin{pmatrix} \cos F \cos\Omega - \sin F \cos I \sin\Omega \\ \cos F \sin\Omega + \sin F \cos I \cos\Omega \\ \sin F \sin I \end{pmatrix}$$

Corrected Precession

- Approx. 1: Zero Prec. Angle, Const. Inclination
- Approx. 2: Removing Short-Period Terms by Averaging Method

$$\left\langle \frac{\mathrm{d}\psi}{\mathrm{dt}} \right\rangle = \kappa \cos\theta_0 \left[-\left(1 - \frac{3}{2}\sin^2 I_0\right) + \frac{\sin 2I_0}{\tan 2\theta_0}\cos\Omega + \frac{\sin^2 I_0}{2}\cos 2\Omega \right]$$

$$\left\langle \frac{\mathrm{d}\theta}{\mathrm{dt}} \right\rangle = -\kappa \cos\theta_0 \left[\frac{\sin 2I_0}{2}\sin\Omega + \frac{\tan\theta_0 \sin^2 I_0}{2}\sin 2\Omega \right]$$

- Prec. Const. Correction

$$p = \kappa \cos\theta_0 \left(1 - \frac{3}{2}\sin^2 I_0\right)$$

Averaging Method

- = Removal of Short-Periodic Terms
- Meaning: Medium-scale Time Average
- Step 1: Fourier Expansion w.r.t. Fast Angles
- Step 2: Trig. F. of Fast Angles = 0

$$\langle \sin jF \rangle = 0 \qquad \langle \cos jF \rangle = 0 \quad \text{when} \quad j \neq 0$$

- Ex.: F = Fast Angle

$$\langle r_Z^2 \cos\theta \rangle = \langle \bar{r}^2 \sin^2 F \sin^2 I \cos\theta \rangle = \frac{\bar{r}^2 \sin^2 I_0 \cos\theta_0}{2}$$

Long Period Nutation

- Assuming Const. Speed of Ascending Node
  - Note: Clockwise Motion $n_\Omega < 0$

$$\Delta^{(1)}\psi = \frac{\kappa \cos\theta_0}{n_\Omega}\left[\frac{\sin 2I_0}{\tan 2\theta_0}\sin\Omega + \frac{\sin^2 I_0}{4}\sin 2\Omega\right]$$

$$\Delta^{(1)}\theta = \frac{\kappa \cos\theta_0}{n_\Omega}\left[\frac{\sin 2I_0}{2}\cos\Omega + \frac{\tan\theta_0 \sin^2 I_0}{4}\cos 2\Omega\right]$$

- Long Period = Large Amplitude
  - So-called Nutation: 18.6y

Luni-solar Precession (2)

- Note: Negative $\psi \equiv -\psi^{(1)} = pt, \ \varepsilon \equiv -\theta^{(1)} = \varepsilon_0$
- Moon (without prime), Sun (with prime)
- Obliquity $\varepsilon_0 \approx 84381'', \ \cos\varepsilon_0 \approx 0.917482, \ \sin\varepsilon_0 \approx 0.397777$
- Orbit Inclination $I_0 \approx 18467'', \ \sin I_0 \approx 0.089411$
- Estimate of Luni-solar Precession

$$\kappa \cos\varepsilon_0 \left(1 - \frac{3}{2}\sin^2 I_0\right) \approx 33.7418''/\text{jy}$$

$$\kappa' \cos\varepsilon_0 \approx 15.9954''/\text{jy}$$

$$p \approx 49.7372''/\text{jy}$$

Long P. Nutation (2)

- Note: Negative $\Delta\psi = -\Delta^{(1)}\psi,\ \Delta\varepsilon = -\Delta^{(1)}\theta$
- Moon's Effect Only

$$\Delta\psi = \frac{\kappa\cos\varepsilon_0}{|n_\Omega|}\left[-\left(\frac{\sin 2I}{\tan 2\varepsilon_0}\right)\sin\Omega + \frac{\sin^2 I}{4}\sin 2\Omega\right]$$

$$\Delta\varepsilon = \frac{\kappa\cos\varepsilon_0}{|n_\Omega|}\left[\frac{\sin 2I}{2}\cos\Omega - \frac{\tan\varepsilon_0 \sin^2 I}{4}\cos 2\Omega\right]$$

Long P. Nutation (3)

- Speed of Nodal Motion

$$n_\Omega \approx -6.96289\times 10^4 \text{''/jy} \qquad \frac{\kappa \cos\varepsilon_0}{|n_\Omega|} \approx 103.130''$$

- Amplitude of LP Nutation (Q: Confirm)

$$\psi_\Omega = \frac{-\kappa\cos\varepsilon_0}{|n_\Omega|}\left(\frac{\sin 2I}{\tan 2\varepsilon_0}\right) = -17.202'' \qquad \varepsilon_\Omega = \frac{\kappa\cos\varepsilon_0}{|n_\Omega|}\left(\frac{\sin 2I}{2}\right) = 9.184''$$

$$\psi_{2\Omega} = \frac{\kappa\cos\varepsilon_0}{|n_\Omega|}\left(\frac{\sin^2 I}{4}\right) = 0.206'' \qquad \varepsilon_{2\Omega} = \frac{-\kappa\cos\varepsilon_0}{|n_\Omega|}\left(\frac{\tan\varepsilon_0 \sin^2 I}{4}\right) = -0.089''$$

- Summary

$$\begin{cases} \Delta\psi = -17.202''\sin\Omega + 0.206''\sin 2\Omega \\ \Delta\varepsilon = +9.184''\sin\Omega - 0.089''\sin 2\Omega \end{cases}$$

Short Period Nutation

- Simplification of Expression
  - L→L＋ψ : Adding Precession A. To Mean Long.
  - = Measuring Longitude from Moving Equinox

$$\delta\psi = \cos\varepsilon_0 \left[-N_{2L}\sin 2L + N_\ell \sin\ell - N_{2L+\ell}\sin(2L+\ell) + N_{2L-\ell}\sin(2L-\ell) \right.$$
$$\left. -N_{2L'}\sin 2L' + N_{\ell'}\sin\ell' - N_{2L'+\ell'}\sin(2L'+\ell') + N_{2L'-\ell'}\sin(2L'-\ell') \right]$$

$$\delta\varepsilon = \sin\varepsilon_0 \left[N_{2L}\cos 2L + N_{2L+\ell}\cos(2L+\ell) - N_{2L-\ell}\cos(2L-\ell) \right.$$
$$\left. +N_{2L'}\cos 2L' + N_{2L'+\ell'}\cos(2L'+\ell') - N_{2L'-\ell'}\cos(2L'-\ell') \right]$$

Short P. Nutation (2)

- Constants (Moon=Unprimed, Sun=Primed)

$n_\ell \approx 1.718 \times 10^7 \text{ "/y}$ $n_{\ell'} \approx 1.296 \times 10^6 \text{ "/y}$ $e_M \approx 0.0549, e_E \approx 0.0167$

- Amplitude Coefficient $N_{2L} = \dfrac{\kappa}{2n_\ell}, N_\ell = \dfrac{3\kappa e}{n_\ell}, N_{2L+\ell} = \dfrac{7\kappa e}{6n_\ell}, N_{2L-\ell} = \dfrac{\kappa e}{2n_\ell}$

- Numerical Evaluation (Q: Confirm)

$N_{2L} \approx 0.228", N_\ell \approx 0.075", N_{2L+\ell} \approx 0.029", N_{2L-\ell} \approx 0.013"$

$N_{2L'} \approx 1.383", N_{\ell'} \approx 0.139", N_{2L'+\ell'} \approx 0.054", N_{2L'-\ell'} \approx 0.023"$

Short P. Nutation (3)

- Summary (Q: Compare with Latest Theory)

$$\delta\psi = -1.269''\sin 2L' - 0.209''\sin 2L + 0.128''\sin \ell' + 0.069''\sin \ell - 0.050''\sin\left(2L'+\ell'\right) - 0.027''\sin\left(2L+\ell\right) + 0.021''\sin\left(2L'-\ell'\right) + 0.012''\sin\left(2L-\ell\right)$$

$$\delta\varepsilon = 0.550''\cos 2L' + 0.091''\cos 2L + 0.021''\cos\left(2L'+\ell'\right) + 0.012''\cos\left(2L+\ell\right) - 0.009''\cos\left(2L'-\ell'\right) - 0.005''\cos\left(2L-\ell\right)$$

  - Analytical Dynamics is Needed for Detailed Eval.

A<B<C

- Analytical Treatment is Quite Difficult
- Post-Poisson Approx. $\omega_A, \omega_B \ll \omega_C$
  - Eq. of Rotation for Tidal Torque

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix} + \begin{pmatrix}\alpha\omega_B\omega_C \\ \beta\omega_C\omega_A \\ 0\end{pmatrix} = \frac{3\mu}{r^5}\begin{pmatrix}\alpha r_B r_C \\ \beta r_C r_A \\ \gamma r_A r_B\end{pmatrix}$$

Cassini Approximation

- Pert. Body: Inclined Circular Orbit Again
- Cassini's Laws Hold (Ex.: Moon)

$$\langle \psi \rangle = \Omega, \quad \langle \theta \rangle = -\theta_0, \quad \langle \phi + \psi \rangle = L + \pi$$

- Assuming Small Inclination: I and θ
- Approx. Expr.
  (Q: Show)

$$F = L - \Omega$$

$$\begin{pmatrix} r_A \\ r_B \\ r_C \end{pmatrix} = \bar{r} \begin{pmatrix} 1 \\ 0 \\ -(\theta_0 + I)\sin F \end{pmatrix}$$

Cassini Approx. (2)

- Approximated Eq. of Rotation

$$\frac{d}{dt}\begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix} + \omega_C \begin{pmatrix}\alpha\omega_B \\ \beta\omega_A \\ 0\end{pmatrix} = \frac{-3\mu\beta\left(\theta_0 + I\right)}{\bar{r}^3}\begin{pmatrix}0 \\ \sin F \\ 0\end{pmatrix}$$

- An Integral Exists
  - C-axis Comp. of Angular Velocity = Mean Orbital Longitude Motion $\omega_C = n_L$

Cassini Approx. (3)

- A- & B-Comp.: Eq. of Forced Oscillation

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}\omega_A\\ \omega_B\end{pmatrix}+n_L\begin{pmatrix}\alpha\omega_B\\ \beta\omega_A\end{pmatrix}=W\begin{pmatrix}0\\ \sin F\end{pmatrix}$$

$$W\equiv\frac{-3\mu\beta\left(\theta_0+I\right)}{\bar{r}^3}$$

- Special Solution (Q: Confirm)

$$\begin{pmatrix}\omega_A\\ \omega_B\end{pmatrix}=\frac{W}{n_F^2-n_E^2}\begin{pmatrix}\alpha n_L\sin F\\ -n_F\cos F\end{pmatrix}$$

$$n_E\equiv n_L\sqrt{-\alpha\beta}\ll n_L$$

$$n_F\equiv\frac{\mathrm{d}F}{\mathrm{dt}}=n_L-n_\Omega>n_L$$

  - Q: Obtain General Sol. including Free Osc.

Cassini Approx. (4)

- Approx. Solution of Euler Angles

$$\psi = \psi_0 + \frac{W}{2\theta_0\left(n_F^2 - n_E^2\right)}\left[\left(n_F - \alpha n_L\right)\left(t - t_0\right) - \left(\frac{n_F + \alpha n_L}{2n_F}\right)\sin 2F\right]$$

$$\theta = -\theta_0 + \frac{W}{2\left(n_F^2 - n_E^2\right)}\left(\frac{n_F + \alpha n_L}{2n_F}\right)\cos 2F$$

$$\phi = L + \pi - \psi$$

- Satisfying Two Consistency Conditions

$$\langle \theta \rangle = -\theta_0, \quad \langle \phi + \psi \rangle = L + \pi$$

Cassini Approx. (5)

- Last Consistency Condition $\langle \psi \rangle = \Omega$
  - Determination of Mean Inclination

$$\left\langle \frac{\mathrm{d}\psi}{\mathrm{dt}} \right\rangle = n_\Omega \rightarrow \theta_0 = \frac{n_I}{(-n_\Omega) - n_I} I \qquad n_I \equiv \frac{-3\mu\beta(n_F - \alpha n_L)}{2\bar{r}^3 (n_F^2 - n_E^2)}$$

  - Moon: (Q: Compare with Eckhardt(1981))

$$n_\Omega \approx -6.963 \times 10^4 \text{"/jy}, \; n_F \approx 1.7395 \times 10^7 \text{"/jy}, \; n_L \approx 1.7325 \times 10^7 \text{"/jy}$$

$$\alpha_M = 3.99 \times 10^{-4}, \; \beta_M = -6.32 \times 10^{-4}, \; n_E = 8.70 \times 10^{-3}$$

$$\mu / \bar{r}^3 \approx n_F^2 \qquad n_I \approx 1.648 \times 10^4 \text{"/ jy} \qquad I \approx 18467\text{"} \qquad \therefore \theta_0 \approx 5527\text{"}$$

Post-Cassini Approx.

- Extension of Cassini Approximation

$$\langle \psi \rangle = \Omega, \langle \theta \rangle = -\theta_0, \langle \phi \rangle = \langle v \rangle + \pi$$

  - Argument of Latitude v=ω+f, True Anomaly f
  - Assuming Small Orb. Incl. I, and Incl. θ
  - Circular Orbit
- Approx. Expr. (Q: Show)

$$\begin{pmatrix} r_A \\ r_B \\ r_C \end{pmatrix} \approx \bar{r} \begin{pmatrix} 1 \\ \psi + \phi - (v + \Omega + \pi) \\ -(I - \theta)\sin v \end{pmatrix}$$

Post-Cassini Approx. (2)

- Approximate Equation of Rotation

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}\omega_A\\ \omega_B\\ \omega_C\end{pmatrix}+\omega_C\begin{pmatrix}\alpha\omega_B\\ \beta\omega_A\\ 0\end{pmatrix}=\frac{3\mu}{\bar{r}^3}\begin{pmatrix}0\\ \beta\left(I-\theta\right)\sin v\\ \gamma\left(v+\Omega+\pi-\psi-\phi\right)\end{pmatrix}$$

- θ ~0 → C-axis Comp.: Forced Oscillation

$$\frac{\mathrm{d}^2\left(\psi+\phi\right)}{\mathrm{dt}^2}+n_P^2\left(\psi+\phi\right)=n_P^2\left(v+\Omega+\pi\right)$$ $$n_P\equiv\sqrt{\frac{3\gamma\mu}{\bar{r}^3}}$$

Post-Cassini Approx. (3)

- Libration in Longitude (=Diff. from L)
  - Optical (=Apparent) Libr. $v+\Omega-L$
  - Physical (=Real) Libr. $\tau \equiv \psi+\phi-L-\pi$
- Eq. of Forced Oscillation (Q: Show)
  - Angular Velocity of Free Oscillation: n_P
  - RHS is derived from Orbital Theory

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}t^2}+n_P^2\tau=n_P^2\left(v+\Omega-L\right)=n_P^2\sum_k H_k \sin\left(n_k t+\varphi_k\right)$$

Post-Cassini Approx. (4)

- Solution of Physical Libration in Long.
  - Suffix P = Physical
  - Q: Derive

$$\tau = \tau_0 + \tau_P \sin n_P \left(t - t_0\right) + \sum_k \frac{n_P^2}{n_P^2 - n_k^2} H_k \sin\left(n_k t + \varphi_k\right)$$

- Solution of C-axis Comp. of Ang. Vel.

$$\omega_C = n_L + n_P \tau_P \cos n_P \left(t - t_0\right) + \sum_k \frac{n_k n_P^2}{n_P^2 - n_k^2} H_k \cos\left(n_k t + \varphi_k\right)$$

Post-Cassini Approx. (5)

- A-,B-Axis Comp.: $\theta \sim -\theta_0 \rightarrow$ Forced Osc.

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}\omega_A\\ \omega_B\end{pmatrix} + n_L\begin{pmatrix}\alpha\omega_B\\ \beta\omega_A\end{pmatrix} = \begin{pmatrix}0\\ \sum_k J_k \sin\left(n_k t + \varphi_k\right)\end{pmatrix}$$

- General Solution (Q: Confirm)
  - Ang. Vel. of Free Polar Motion $n_E \equiv n_L\sqrt{-\alpha\beta}$

$$\begin{pmatrix}\omega_A\\ \omega_B\end{pmatrix} = \omega_0\begin{pmatrix}\sqrt{\alpha}\cos n_E\left(t-t_0\right)\\ \sqrt{-\beta}\sin n_E\left(t-t_0\right)\end{pmatrix} + \sum_k \frac{J_k}{n_k^2 - n_E^2}\begin{pmatrix}\alpha n_L \sin\left(n_k t + \varphi_k\right)\\ -n_k\cos\left(n_k t + \varphi_k\right)\end{pmatrix}$$

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