A Treatise on Rotation of Celestial Bodies. Part II

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Index (0-6 in Part I)

7. Analytical Dynamics of Rotation

- Two Formalism: Lagrangian vs Hamiltonian
- Generalized Coordinates $\mathbf{q}=(\psi,\theta,\phi)^{\mathrm{T}}$ $\dot{\mathbf{q}}=\frac{\mathrm{d}\mathbf{q}}{\mathrm{dt}}$
- Lagrangian
  - T: Kinetic Energy $\mathcal{L}(\dot{\mathbf{q}},\mathbf{q},t)\equiv T(\dot{\mathbf{q}},\mathbf{q})-U(\mathbf{q},t)$
  - U: Potential Energy, V=－U: Force Function
- Lagrangian Eq. of Motion: Generalized Force **F**

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}\right)_{\mathbf{q}}\right]=\left(\frac{\partial\mathcal{L}}{\partial\mathbf{q}}\right)_{\dot{\mathbf{q}}}+\mathbf{F}\rightarrow\frac{\mathrm{d}}{\mathrm{dt}}\left[\left(\frac{\partial T}{\partial\dot{\mathbf{q}}}\right)_{\mathbf{q}}\right]=\left(\frac{\partial T}{\partial\mathbf{q}}\right)_{\dot{\mathbf{q}}}-\left(\frac{\partial U}{\partial\mathbf{q}}\right)_{\dot{\mathbf{q}}}+\mathbf{F}$$

Angular Velocity and Time Variation of Angle

- Ang. Vel. $\boldsymbol{\omega}$, Time. Var. of Gen. Coord. $\mathbf{q}$

$$\omega_A = \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi, \omega_B = \dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi, \omega_C = \dot{\psi}\cos\theta + \dot{\phi}$$

- Matrix Expression $\boldsymbol{\omega} = \mathbf{Q}(\mathbf{q})\dot{\mathbf{q}}$

$$\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix} = \begin{pmatrix} \sin\theta\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Definition of Kinetic Energy

- Kinetic Energy

$$2T = aL_A^2 + bL_B^2 + cL_C^2 = A\omega_A^2 + B\omega_B^2 + C\omega_C^2$$

- Expr. By Gen. Coord. **q** and its Time Var. **q**

$$2T(\dot{\mathbf{q}}, \mathbf{q}) = \boldsymbol{\omega}^{\mathrm{T}} \mathbf{I} \boldsymbol{\omega} = \dot{\mathbf{q}}^{\mathrm{T}} \mathbf{K}(\mathbf{q}) \dot{\mathbf{q}}$$

- Coefficient Matrix
  - Caution: Depends on Generalized Coordinates

$$\mathbf{K}(\mathbf{q}) \equiv \mathbf{Q}(\mathbf{q})^{\mathrm{T}} \mathbf{I} \mathbf{Q}(\mathbf{q})$$

Expression of Kinetic Energy

- Expression of Coefficient Matrix (Q: Show)

$$\mathbf{K} = \begin{pmatrix} K_{\psi\psi} & K_{\psi\theta} & C\cos\theta \\ K_{\psi\theta} & K_{\theta\theta} & 0 \\ C\cos\theta & 0 & C \end{pmatrix}$$

$$K_{\psi\psi} = C - \left[\frac{2C-(A+B)}{2}\right]\sin^2\theta - \left(\frac{A-B}{2}\right)\sin^2\theta\cos 2\phi$$

$$K_{\psi\theta} = \left(\frac{A-B}{2}\right)\sin\theta\sin 2\phi,\ K_{\theta\theta} = \left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right)\cos 2\phi$$

Angle Partial Deriv. of T

- Vector Expr. of Angle Part. D. $\left(\frac{\partial T}{\partial \mathbf{q}}\right)_{\dot{\mathbf{q}}} = \dot{\mathbf{q}}^{\mathrm{T}}\left(\frac{\partial \mathbf{K}}{\partial \mathbf{q}}\right)\dot{\mathbf{q}}$
- PD by Prec. Angle=0 $\left(\frac{\partial T}{\partial \psi}\right)_{\dot{\mathbf{q}}} = 0$
  - Q: Answer Physical Meaning
- PD by Rotation Angle

  =0 if A=B
  - Q: Why?

$$\left(\frac{\partial \mathbf{K}}{\partial \phi}\right)_{\psi,\theta} = \begin{pmatrix} K_{\psi\psi,\phi} & K_{\psi\theta,\phi} & 0 \\ K_{\psi\theta,\phi} & K_{\theta\theta,\phi} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K_{\psi\psi,\phi} = (A-B)\sin^2\theta\sin 2\phi,\quad K_{\psi\theta,\phi} = (A-B)\sin\theta\cos 2\phi,$$
$$K_{\theta\theta,\phi} = -(A-B)\sin 2\phi$$

Angle Part. D. of T (2)

- Partial Derivative by Inclination Angle

$$\left(\frac{\partial \mathbf{K}}{\partial \theta}\right)_{\psi,\phi} = \begin{pmatrix} K_{\psi\psi,\theta} & K_{\psi\theta,\theta} & -C\sin\theta \\ K_{\psi\theta,\theta} & 0 & 0 \\ -C\sin\theta & 0 & 0 \end{pmatrix}$$

$$K_{\psi\psi,\theta} = \left(\frac{2C - A + B}{4}\right)\sin 2\theta - \left(\frac{A - B}{4}\right)\sin 2\theta \cos 2\phi$$

$$K_{\psi\theta,\theta} = \left(\frac{A - B}{2}\right)\cos\theta \sin 2\phi$$

Expr. of Force Function

- V=-U depends on Basis Matrix of Principal CS

$$V = V(\mathbf{e}_A, \mathbf{e}_B, \mathbf{e}_C)$$

- Ex.: 2nd Order Tidal FF by External Masspoint

$$V_2 = \frac{3\mu}{4r^3}\left[(A+B-2C)n_C^2 + (B-A)(n_A^2 - n_B^2)\right]$$

$$n_A = \mathbf{n}\cdot\mathbf{e}_A, n_B = \mathbf{n}\cdot\mathbf{e}_B, n_C = \mathbf{n}\cdot\mathbf{e}_C, \mathbf{n} = \frac{\mathbf{r}}{r}$$

Expansion of V

- Ex.: Grav. Tidal FF by External Masspoints

$$V = \mu \int \frac{\rho(\mathbf{x})}{|\mathbf{r}-\mathbf{x}|} d^3\mathbf{x} = V_0 + V_1 + V_2 + \cdots$$

$$V_0 = \frac{\mu M}{r}$$

- 0th Order: Indep. On Orientation $V_1 = 0$
- 1st Order =0: Barycenter Def.
- 2nd Order: Expr. By A, B, C
  - MacCullagh's Formula (Q: Derive)

$$\begin{pmatrix} r_A \\ r_B \\ r_C \end{pmatrix} = r \begin{pmatrix} \cos\beta\cos\lambda \\ \cos\beta\sin\lambda \\ \sin\beta \end{pmatrix}$$

$$V_2 = \frac{\mu}{r^3}\left[\left(\frac{A+B}{2} - C\right) P_2(\sin\beta) + \left(\frac{B-A}{4}\right) P_2^2(\sin\beta)\cos 2\lambda\right]$$

$$P_2(x) = \frac{3x^2-1}{2}$$

$$P_2^2(x) = 3(1-x^2)$$

Angle PD Formula of V

- PD by Arbitrary Angle $\frac{\partial V}{\partial \varphi} = \sum_{J=A,B,C} \left(\frac{\partial \mathbf{e}_J}{\partial \varphi} \right) \bullet \left(\frac{\partial V}{\partial \mathbf{e}_J} \right)$
- Applying Vector Exp. of Inf. Rot. $\frac{\partial \mathbf{e}_J}{\partial \varphi} = \mathbf{e}_\varphi \times \mathbf{e}_J$
  - PD of Basis = Vec. Prod. w Rot. Axis
- Relation w Torque $\frac{\partial V}{\partial \varphi} = -\mathbf{e}_\varphi \bullet \mathbf{N}$
- Gen. Def. Torque $\mathbf{N} \equiv -\sum_{J=A,B,C} \mathbf{e}_J \times \left(\frac{\partial V}{\partial \mathbf{e}_J} \right)$
  - When V is a function of Principal CS Comp. of **r**

$$V = V(r_A, r_B, r_C) \rightarrow \mathbf{N} = -\mathbf{r} \times \mathbf{F} \qquad \mathbf{F} \equiv \frac{\partial V}{\partial \mathbf{r}}$$

Torque and PD of V

- Expression in Principal CS (Q: Derive)

$$N_A = -\left(\frac{\partial V}{\partial \theta}\right)_{\psi,\phi} \cos\phi - \left(\frac{\partial V}{\partial \lambda}\right) \sin\phi$$

$$N_B = \left(\frac{\partial V}{\partial \theta}\right)_{\psi,\phi} \sin\phi - \left(\frac{\partial V}{\partial \lambda}\right) \cos\phi$$

$$N_C = -\left(\frac{\partial V}{\partial \phi}\right)_{\psi,\theta}$$

$$\frac{\partial V}{\partial \lambda} \equiv \frac{1}{\sin\theta}\left[\left(\frac{\partial V}{\partial \psi}\right)_{\theta,\phi} - \left(\frac{\partial V}{\partial \phi}\right)_{\psi,\theta} \cos\theta\right]$$

Torque and PD of V (2)

- Expression in Inertial CS (Q: Derive)

$$N_X = -\left(\frac{\partial V}{\partial \theta}\right)_{\psi,\phi} \cos\psi - \left(\frac{\partial V}{\partial \varphi}\right) \sin\psi$$

$$N_Y = -\left(\frac{\partial V}{\partial \theta}\right)_{\psi,\phi} \sin\psi + \left(\frac{\partial V}{\partial \varphi}\right) \cos\psi$$

$$N_Z = -\left(\frac{\partial V}{\partial \psi}\right)_{\theta,\phi}$$

$$\frac{\partial V}{\partial \varphi} \equiv \frac{1}{\sin\theta}\left[\left(\frac{\partial V}{\partial \phi}\right)_{\psi,\theta} - \left(\frac{\partial V}{\partial \psi}\right)_{\theta,\phi} \cos\theta\right]$$

Lagrangian Eq. of Motion

- Rewriting Lagrangian Equation of Motion

$$\frac{\mathrm{d}\left(\mathbf{K}\dot{\mathbf{q}}\right)}{\mathrm{dt}} = \mathbf{K}\frac{\mathrm{d}^2\mathbf{q}}{\mathrm{dt}^2} + \dot{\mathbf{q}}^{\mathrm{T}}\left(\frac{\partial \mathbf{K}}{\partial \mathbf{q}}\right)_{\dot{\mathrm{q}}}\dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\left(\frac{\partial \mathbf{K}}{\partial \mathbf{q}}\right)_{\dot{\mathrm{q}}}\dot{\mathbf{q}} + \left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\dot{\mathrm{q}}} + \mathbf{F}$$

- Transform into 2nd Order Ordinary Diff. Eq.

$$\frac{\mathrm{d}^2\mathbf{q}}{\mathrm{dt}^2} = -\mathbf{K}^{-1}\left[\frac{1}{2}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{dt}}\right)^{\mathrm{T}}\left(\frac{\partial \mathbf{K}}{\partial \mathbf{q}}\right)_{\dot{\mathrm{q}}}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{dt}}\right) - \left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\dot{\mathrm{q}}} - \mathbf{F}\right]$$

- A<B: Complexity due to Non-Diagonal **K**
  - From Lagrangian to Hamiltonian Formalism

Lagrangian EoM (A=B)

- Assuming Rot. Angle Comp. of GF =0 $F_\phi = 0$
  - → Conserv. C-Comp. of AV $\omega_C \equiv \left(\frac{d\phi}{dt}\right) + \left(\frac{d\psi}{dt}\right)\cos\theta$
- Eq. of Motion (Q: Derive)

$$\frac{d^2\psi}{dt^2} = \frac{1}{\sin\theta}\left(\frac{d\theta}{dt}\right)\left[\frac{C\omega_C}{A} - 2\left(\frac{d\psi}{dt}\right)\cos\theta\right] + \frac{1}{A\sin^2\theta}\left[\left(\frac{\partial V}{\partial\psi}\right)_{\theta,\phi} - F_\psi\right]$$

$$\frac{d^2\theta}{dt^2} = -\sin\theta\left(\frac{d\psi}{dt}\right)\left[\frac{C\omega_C}{A} - \left(\frac{d\psi}{dt}\right)\cos\theta\right] + \frac{1}{A}\left[\left(\frac{\partial V}{\partial\theta}\right)_{\psi,\phi} - F_\theta\right]$$

$$\frac{d\phi}{dt} = \omega_C - \left(\frac{d\psi}{dt}\right)\cos\theta$$

Poisson Approx. EoM

- Poisson Approx.
  - Ignore Small Quant.
  - Q: Explain its Physical Meaning
- Approximated Eq. of Motion
  - Q: Derive

$$\left|\frac{d^2\psi}{dt^2}\right|, \left|\frac{d^2\theta}{dt^2}\right|, \left|\frac{d\psi}{dt}\right|\left|\frac{d\theta}{dt}\right|, \left|\frac{d\psi}{dt}\right|^2 \approx 0$$

$$\frac{d\psi}{dt} = \frac{1}{C\omega_C \sin\theta}\left[\left(\frac{\partial V}{\partial \theta}\right)_{\psi,\phi} - F_\theta\right]$$

$$\frac{d\theta}{dt} = \frac{-1}{C\omega_C \sin\theta}\left[\left(\frac{\partial V}{\partial \psi}\right)_{\theta,\phi} - F_\psi\right]$$

$$\frac{d\phi}{dt} = \omega_C - \left(\frac{d\psi}{dt}\right)\cos\theta$$

Generalized Momentum

- Definition
$$\mathbf{p} \equiv \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right)_{\mathbf{q}} = \left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)_{\mathbf{q}} = \mathbf{K}\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)$$
  - Be Careful when **I** = **I**(d**q**/dt) as Non-rigid Body
- Generalized Momentum w.r.t. Euler Angles
$$\mathbf{p} = \begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} K_{\psi\psi}\dot{\psi} + K_{\psi\theta}\dot{\theta} + (C\cos\theta)\dot{\phi} \\ K_{\psi\theta}\dot{\psi} + K_{\theta\theta}\dot{\theta} \\ (C\cos\theta)\dot{\psi} + C\dot{\phi} \end{pmatrix}$$

Gen. Momentum (2)

- Expression of Angular Momentum

$$\begin{pmatrix} L_A \\ L_B \\ L_C \end{pmatrix} = \begin{pmatrix} p_\theta \cos\phi + p_\lambda \sin\phi \\ -p_\theta \sin\phi + p_\lambda \cos\phi \\ p_\phi \end{pmatrix} \qquad p_\lambda \equiv \frac{p_\psi - p_\phi \cos\theta}{\sin\theta}$$

- Kinetic Energy: Still Complicated (Q: Show)

$$4T = 2cp_\phi^2 + (a+b)\left(p_\theta^2 + p_\lambda^2\right) + (a-b)\left[\left(p_\theta^2 - p_\lambda^2\right)\cos 2\phi + 2 p_\theta p_\lambda \sin 2\phi\right]$$

Hamiltonian

- Assuming Gen. Force **F**=0 (i.e. No Dissipation)
- Definition of Hamiltonian

$$\mathcal{H} \equiv \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{p} - \mathcal{L} = T + U = T - V$$

- Hamiltonian Equation of Motion

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{dt}} = -\left(\frac{\partial \mathcal{H}}{\partial \mathbf{q}}\right)_{\mathbf{p}} = -\left(\frac{\partial T}{\partial \mathbf{q}}\right)_{\mathbf{p}} + \left(\frac{\partial V}{\partial \mathbf{q}}\right)_{\mathbf{p}}$$

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{dt}} = \left(\frac{\partial \mathcal{H}}{\partial \mathbf{p}}\right)_{\mathbf{q}} = \mathbf{K}^{-1}\mathbf{p}$$

Angle PD of T (3)

- Caution: Difference in Fixed Var. in PD Eval.
- Euler Angle: Yet Complicated Expression

$$\left(\frac{\partial T}{\partial \psi}\right)_{\mathbf{p},\theta,\phi} = 0 \qquad \left(\frac{\partial T}{\partial \phi}\right)_{\mathbf{p},\psi,\theta} = \left(\frac{a-b}{2}\right)\left[-\left(p_\theta^2 - p_\lambda^2\right)\sin 2\phi + 2 p_\theta p_\lambda \cos 2\phi\right]$$

$$\left(\frac{\partial T}{\partial \theta}\right)_{\mathbf{p},\psi,\phi} = \left[\left(\frac{a+b}{2}\right)p_\lambda - \left(\frac{a-b}{2}\right)\left(p_\lambda \cos 2\phi - p_\theta \sin 2\phi\right)\right]\left(\frac{\partial p_\lambda}{\partial \theta}\right)$$

$$\mathbf{p} = \left(p_\psi, p_\theta, p_\phi\right)^{\mathrm{T}} \qquad p_\lambda \equiv \frac{p_\psi - p_\phi \cos\theta}{\sin\theta} \qquad \left(\frac{\partial p_\lambda}{\partial \theta}\right)_{\mathbf{p},\psi,\phi} = \frac{p_\phi - p_\psi \cos\theta}{\sin^2\theta}$$

Serret Canonical Variable

- Also called as Andoyer Canical Variable
- Canonical Transform of Serret (1866)

$$\left(p_\psi, p_\theta, p_\phi; \psi, \theta, \phi\right) \to \left(L \equiv p_\phi, G \equiv \left|\mathbf{L}\right|, H \equiv p_\psi; \ell, g, h\right)$$

- Generating Function of Can. Transf.

$$S \equiv \int p_\psi d\psi + \int p_\theta d\theta + \int p_\phi d\phi = L\phi + H\psi + \int p_\theta d\theta$$

$$p_\theta = -\sqrt{G^2 - L^2 - \left(\frac{H - L\cos\theta}{\sin\theta}\right)^2}$$

Serret Can. Angle Var.

- Expression of Serret Angle Variable
  - Definition by Gen. Function → Complicated

$$\ell \equiv \left(\frac{\partial S}{\partial L}\right)_{G,H,\phi,\theta,\psi} = \phi - \int \frac{\left(L\cos 2\theta - H\cos\theta\right)\mathrm{d}\theta}{\sin\theta\sqrt{G^2\sin^2\theta - H^2 + 2HL\cos\theta - L^2}}$$

$$g \equiv \left(\frac{\partial S}{\partial G}\right)_{L,H,\phi,\theta,\psi} = -\int \frac{G\sin\theta\,\mathrm{d}\theta}{\sqrt{G^2\sin^2\theta - H^2 + 2HL\cos\theta - L^2}}$$

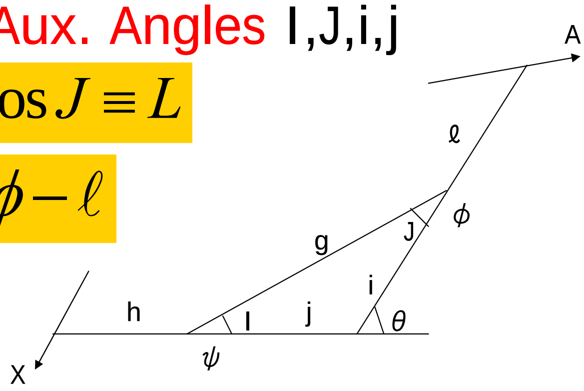
$$h \equiv \left(\frac{\partial S}{\partial H}\right)_{L,G,\phi,\theta,\psi} = \psi - \int \frac{\left(L\cos\theta - H\right)\mathrm{d}\theta}{\sin\theta\sqrt{G^2\sin^2\theta - H^2 + 2HL\cos\theta - L^2}}$$

Auxiliary Angles

- Introduction of Aux. Angles I,J,i,j

$$G\cos I \equiv H,\ G\cos J \equiv L$$

$$j \equiv \psi - h,\quad i \equiv \phi - \ell$$

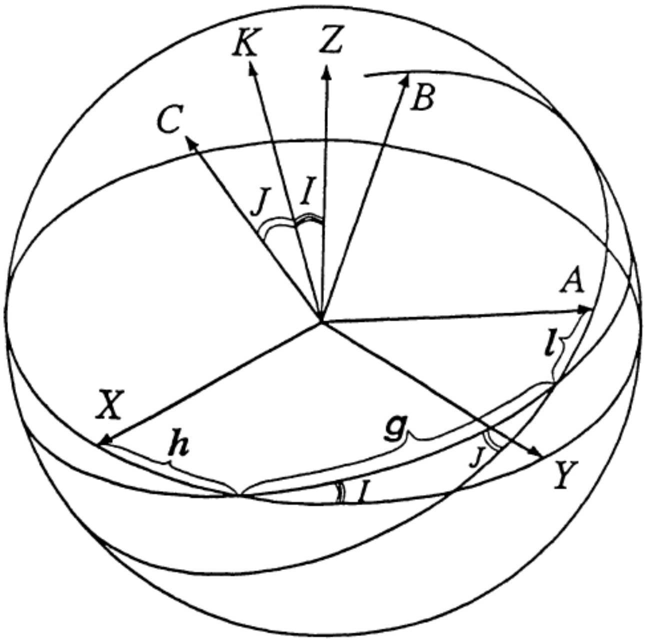


- Geometric Def.
  - Q: Prove
- Analytical Expression?

$$j = j(I,g,J),\theta = \theta(I,g,J),i = i(I,g,J)$$

Aux. Angle (2)

- Geometric Relation



Aux. Angle (3)

- Matrix Relation (Q: Derive from Geom.)

$$\mathbf{R}_3(i)\mathbf{R}_1(\theta)\mathbf{R}_3(j)=\mathbf{R}_1(J)\mathbf{R}_3(g)\mathbf{R}_1(I)$$

- Spher. Triangle Formula from Matrix Rel.
  - 2nd Cosine F. $\cos\theta=\cos I\cos J-\sin I\sin J\cos g$
  - No-name Formula (J,j⇔I,i) (Q: Derive)

$$\tan i=\frac{\sin g\sin I}{\sin J\cos I+\cos J\sin I\cos g}$$

Aux. Angle (4)

- Euler Angle expressed by Aux. Angle

$$\psi = h + j = h + \tan^{-1}\left(\frac{\sin g \sin J}{\sin I \cos J + \cos I \sin J \cos g}\right)$$

$$\theta = \cos^{-1}\left(\cos I \cos J - \sin I \sin J \cos g\right)$$

$$\phi = \ell + i = \ell + \tan^{-1}\left(\frac{\sin g \sin I}{\sin J \cos I + \cos J \sin I \cos g}\right)$$

Case I»J~0

- Ex.: Earth Rotation in Ecliptic CS
  - Obliquity of Ecliptic: I ~ 23.5 deg
  - Polar Motion: J ~ 0.2 arcsec ~ 10^{-6} radian
- Variation of No-name Formula (Q: Show)

$$i = g - \tan^{-1}\left(\frac{\sin J \cot I \sin g - \sin^2(J/2)\sin 2g}{1 + \sin J \cot I \cos g - \sin^2(J/2)(1+\cos 2g)}\right)$$

Case I»J~0 (2)

- Variation of 2nd Cosine Formula (Q: Show)

$$\theta = I + \tan^{-1}\left[\frac{\Delta C\left(2-\Delta C\cos I-\Delta S\sin I\right)}{\left(2\sin I-\Delta S\right)\left(1-\Delta C\cos I-\Delta S\sin I\right)}\right]$$

$$\Delta C \equiv \cos I - \cos\theta = \sin J\sin I\cos g + \frac{1}{2}\sin^2 J\cos I$$

$$\Delta S \equiv \sin I - \sin\theta = \frac{-\Delta C\left(2\cos I-\Delta C\right)}{\sin I+\sqrt{\sin^2 I+\Delta C\left(2\cos I-\Delta C\right)}}$$

Case I»J~0 (3)

- Approximate Expression of Euler Angles
  - Q: Derive from Variation of Formulas

$$\psi \cong h + \left(\frac{J}{\sin I}\right)\sin g - \frac{J^2}{2}\cot I \sin 2g + O\left(J^3\right)$$

$$\theta \cong I + J\cos g + \frac{J^2}{2}\cot I \sin^2 g + O\left(J^3\right)$$

$$\phi \cong \ell + g - J\cot I \sin g + \frac{J^2}{2}\left(\cot^2 I + \frac{1}{2}\right)\sin 2g + O\left(J^3\right)$$

Kinetic Energy

- Canonical Expression of Kinetic Energy

$$4T = (a+b)G^2 - (a+b-2c)L^2 - (a-b)(G^2 - L^2)\cos 2\ell$$

  - Averaged by ℓ → Hyperbolic w.r.t G and L
- Note: T is Independent on H, g, and h

$$\left(\frac{\partial T}{\partial H}\right)_{L,G,\ell,g,h} = \left(\frac{\partial T}{\partial g}\right)_{L,G,H,\ell,h} = \left(\frac{\partial T}{\partial h}\right)_{L,G,H,\ell,g} = 0$$

Partial Derivative of T

- Using Serret Canonical Variable: Simple

$$\left(\frac{\partial T}{\partial L}\right)_{G,H,\ell,g,h} = -L\left[\left(\frac{a+b}{2}-c\right)-\left(\frac{a-b}{2}\right)\cos 2\ell\right]$$

$$\left(\frac{\partial T}{\partial G}\right)_{L,H,\ell,g,h} = G\left[\left(\frac{a+b}{2}\right)-\left(\frac{a-b}{2}\right)\cos 2\ell\right]$$

$$\left(\frac{\partial T}{\partial \ell}\right)_{L,G,H,g,h} = \left(\frac{a-b}{2}\right)\left(G^2-L^2\right)\sin 2\ell$$

Canonical Eq. of Motion

- Hamilton's Canonical Equation of Motion
  - Action Var.: L, G, H Angle Var.: ℓ, g, h

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\left(\frac{\partial \mathcal{H}}{\partial \ell}\right)_{L,G,H,g,h} \qquad \frac{\mathrm{d}\ell}{\mathrm{d}t} = +\left(\frac{\partial \mathcal{H}}{\partial L}\right)_{G,H,\ell,g,h}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = -\left(\frac{\partial \mathcal{H}}{\partial g}\right)_{L,G,H,\ell,h} \qquad \frac{\mathrm{d}g}{\mathrm{d}t} = +\left(\frac{\partial \mathcal{H}}{\partial G}\right)_{L,H,\ell,g,h}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\left(\frac{\partial \mathcal{H}}{\partial h}\right)_{L,G,H,\ell,g} \qquad \frac{\mathrm{d}h}{\mathrm{d}t} = +\left(\frac{\partial \mathcal{H}}{\partial H}\right)_{L,G,\ell,g,h}$$

Canon. Eq. of Motion (2)

- Using Serret Canonical Variables
  - Variable of Small Variation: G, H, h
  - Variable of Large Variation: L, ℓ, g

$$\frac{\mathrm{d}G}{\mathrm{dt}} = \left(\frac{\partial V}{\partial g}\right)_{L,G,H,\ell,h}$$

$$\frac{\mathrm{d}H}{\mathrm{dt}} = \left(\frac{\partial V}{\partial h}\right)_{L,G,H,\ell,g}$$

$$\frac{\mathrm{d}h}{\mathrm{dt}} = -\left(\frac{\partial V}{\partial H}\right)_{L,G,\ell,g,h}$$

$$\frac{\mathrm{d}L}{\mathrm{dt}} = -\left(\frac{a-b}{2}\right)\left(G^2 - L^2\right)\sin 2\ell + \left(\frac{\partial V}{\partial \ell}\right)_{L,G,H,g,h}$$

$$\frac{\mathrm{d}\ell}{\mathrm{dt}} = -L\left[\left(\frac{a+b}{2} - c\right) - \left(\frac{a-b}{2}\right)\cos 2\ell\right] - \left(\frac{\partial V}{\partial L}\right)_{G,H,\ell,g,h}$$

$$\frac{\mathrm{d}g}{\mathrm{dt}} = G\left[\left(\frac{a+b}{2}\right) - \left(\frac{a-b}{2}\right)\cos 2\ell\right] - \left(\frac{\partial V}{\partial G}\right)_{L,H,\ell,g,h}$$

Torque-Free Solution

- V=0 → Trivial Solution $G = G_0, H = H_0, h = h_0$
- Basic Eq.: 2 Essential Freedom (Q: Why?)

$$\frac{\mathrm{d}L}{\mathrm{dt}} = -\left(\frac{a-b}{2}\right)\left(G_0^2 - L^2\right)\sin 2\ell$$

$$\frac{\mathrm{d}\ell}{\mathrm{dt}} = -L\left[\left(\frac{a+b}{2} - c\right) - \left(\frac{a-b}{2}\right)\cos 2\ell\right]$$

$$\frac{\mathrm{d}g}{\mathrm{dt}} = G_0\left[\left(\frac{a+b}{2}\right) - \left(\frac{a-b}{2}\right)\cos 2\ell\right]$$

Torque-Free Solution (2)

- Case A=B (i.e., a=b): Simple
  - L, G, H, and h are Constant

$$L = L_0,\ G = G_0,\ H = H_0,\ h = h_0$$

  - ℓ and g are Linear Function of Time

$$\ell = \ell_0 + n_\ell (t - t_0),\ g = g_0 + n_g (t - t_0)$$

  - ℓ: Reverse and Slow Motion, g: Fast Motion

$$n_\ell = -(a - c) L_0 < 0,\ n_g = a G_0 > 0$$

Equal Energy Curve

- Case A<B → G=Const. → Freedom 2
- Normalized Kinetic Energy

$$E \equiv \frac{4T - 2cG^2}{(a+b-2c)G^2} = \sin^2 J (1 - Q\cos 2\ell)$$

$$Q \equiv \frac{a-b}{a+b-2c}$$

$$0 \le Q \le 1$$

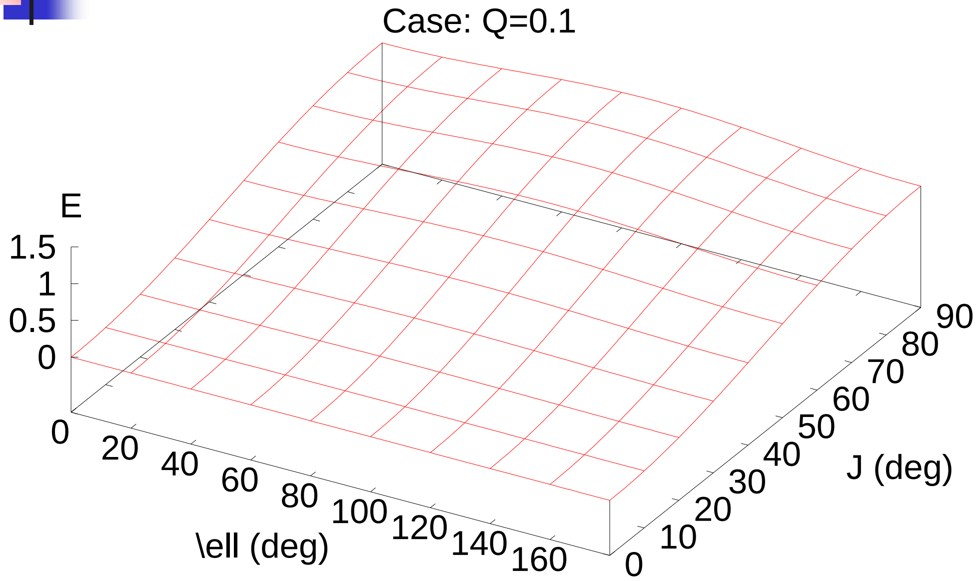
$$Q_{\mathrm{Earth}} = 3.334 \times 10^{-4}$$

$$Q_{\mathrm{Moon}} = 0.226$$

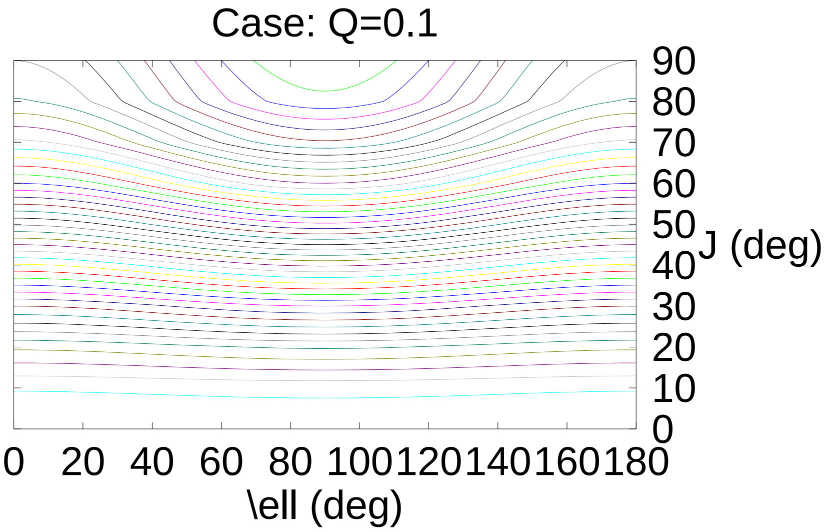
$$0 \le E \le 1 + Q$$

- Equal Energy Curve
  - Mode Changes w.r.t. Energy
  - Libration (E>1-Q) vs Circulation (E<1-Q)
  - A~B → Q~0 → Circulation is Major

Normalized Energy



Equal Energy Curve (2)



Torque-Free Solution (3)

- Case A<B<C (Rigorous Treatment) $a > b > c$
  - Energy Integral (Q: Show its conservation)

$$4T_0 \equiv 2G_0^2 d = (a+b)G_0^2 - (a+b-2c)L^2 - (a-b)(G_0^2 - L^2)\cos 2\ell$$

$$a \ge d \ge c$$

  - Deleting ℓ (Q: Derive)

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \pm\sqrt{(a-c)(b-c)(L_0^2 - L^2)(L^2 - W_1)}$$

$$L_0 \equiv \sqrt{\frac{a-d}{a-c}} G_0 \le G_0 \qquad W_1 \equiv \left(\frac{b-d}{b-c}\right) G_0^2 \le L_0^2$$

Solution Classification

- 5 Solution Class: w.r.t. Value of d
  - Largest: d=a, k→∞ ... Uniform Rot. Around A-axis
  - Large: d>b, k>1 ... Long- (or A-) Axis Mode
  - Critical: d=b, k=1 ... Asympt. Rot. Around B-Axis
  - Small: d<b, k<1 ... Short- (or C-) Axis Mode
  - Smallest: d=c, k=0 ... Uniform Rot. Around C-Axis
- Similarity and Difference w.r.t. Kepler Motion
  - Similarity: Eccentricity, e ⇔ Modulus, k
  - Difference: Period Length (Infinite only when k=1)

Low Energy: d<b

- Domain of L + Transformation to Phase Angle

$$L_1 \equiv \sqrt{W_1} \le L \le L_0 \qquad L = L_0\sqrt{1-k_L^2 \sin^2 \varphi_L} \qquad k_L^2 \equiv \frac{L_0^2 - L_1^2}{L_0^2} < 1$$

- Eq. of Phase Angle and Solution (Q: Derive)

$$\frac{\mathrm{d}\varphi_L}{\mathrm{d}u_L} = \sqrt{1-k_L^2 \sin^2 \varphi_L}, \left(\varphi_L = 0 \text{ at } u_L = 0\right) \to \varphi_L = \mathrm{am}\left(u_L; k_L\right)$$

- Argument and Modulus of Elliptic Functon

$$u_L \equiv G_0\sqrt{(a-d)(b-c)}\,(t-t_0) = L_0\sqrt{(a-c)(b-c)}\,(t-t_0) \qquad k_L = \sqrt{\frac{(a-b)(d-c)}{(a-d)(b-c)}}$$

Low Energy (2)

- Solution using Elliptic Function
  - Q: Confirm Equality with Solution in Section 4

$$L = L_0 \mathrm{dn}(u_L; k_L)$$

$$n \equiv \frac{a-b}{b-c}$$

$$n_{\mathrm{Earth}} = 6.668 \times 10^{-4}$$

$$n_{\mathrm{Moon}} = 0.584$$

$$\ell = \frac{\pi}{2} - \mathrm{am}(u_L; k_L) - \tan^{-1}\left[\frac{n \mathrm{sn}(u_L; k_L) \mathrm{cn}(u_L; k_L)}{1 + \sqrt{1+n} + n \mathrm{sn}^2(u_L; k_L)}\right]$$

$$g = g_0 + cG_0(t - t_0) + \frac{G_0\sqrt{1+n}}{L_0} \mathrm{pn}(u_L; n, k_L)$$

Low Energy (3)

- Solution using Amplitude Function
  - Q: Derive $\varphi = \mathrm{am}\left(u_L; k_L\right)$

$$L = L_0\sqrt{1 - k_L^2 \sin^2\varphi}$$

$$\ell = \frac{\pi}{2} - \varphi - \tan^{-1}\left[\frac{n\sin\varphi\cos\varphi}{1 + \sqrt{1+n} + n\sin^2\varphi}\right]$$

$$g = g_0 + cG_0\left(t - t_0\right) + \frac{G_0\sqrt{1+n}}{L_0}\Pi\left(\varphi; n, k_L\right)$$

Low Energy (4)

- Case A=B: Then a=b, n=0, k=0
  - Q: Confirm Equality with Solution in Section 4 by Setting $\ell_0 = \pi/2$

$$L = L_0$$

$$\ell = \frac{\pi}{2} - u \qquad u = (a-c)L_0(t-t_0)$$

$$g = g_0 + aG_0(t-t_0)$$

Low Energy (5)

- Case of Earth $n_{\text{Earth}} = 6.668\times10^{-4}, k_{\text{Earth}} \sim 3\times10^{-8}$
  - Not n but m=k^2 is negligible

$$L \cong L_0$$

$$\ell \cong \frac{\pi}{2} - u - \tan^{-1}\left[\frac{n\sin u\cos u}{1+\sqrt{1+n}+n\sin^2 u}\right]$$

$$g \cong g_0 + cG_0\left(t-t_0\right) + \frac{G_0}{L_0}\tan^{-1}\left(\sqrt{1+n}\tan u\right)$$

Critical Energy: d=b

- Domain of L + Variable Transformation

$$0 \le L \le L_0 \qquad L = \frac{L_0}{\cosh \varphi_C}$$

- Transformed Eq. of Motion and Solution

$$\frac{\mathrm{d}\varphi_C}{\mathrm{d}u_C} = 1 \to \varphi_C = u_C$$

- Expression of Argument

$$u_C \equiv G_0 \sqrt{(a-b)(b-c)}\,(t-t_0)$$

Critical Energy (2)

- Solution using Hyperbolic Function
  - Q: Derive it by Taking Limit of k=1 in Low Energy Solution

$$L=\frac{L_0}{\cosh u_C}$$

$$\ell=\frac{\pi}{2}-\tan^{-1}\left[\frac{\sinh u_C}{\sigma}\right]$$

$$g = g_0+cG_0\left(t-t_0\right)+\frac{a-c}{\sqrt{(a-b)(b-c)}}\Pi_x\left(\sinh u_C;n,1\right)$$

High Energy: d>b

- Transformation to Phase Angle $k_H \equiv \frac{1}{k_L} < 1$

  $-L_0 \le L \le L_0$ $\quad L = L_0 \cos\varphi_H$

- Eq. of Phase Angle and Solution (Q: Derive)

  $$\frac{\mathrm{d}\varphi_H}{\mathrm{du}_H} = \sqrt{1-k_H^2 \sin^2\varphi_H} \to \varphi_H = \mathrm{am}\left(u_H; k_H\right) \quad u_H \equiv \frac{k_L u_L}{k_H}$$

- Argument and Modulus of Elliptic Functon

  $$u_H \equiv G_0\sqrt{(a-b)(d-c)}\,(t-t_0) \quad k_H = \sqrt{\frac{(a-d)(b-c)}{(a-b)(d-c)}}$$

High Energy (2)

- Solution using Elliptic Function
  - Q: Derive from Low Energy Solution

  by Reciprocal Transformation of Modulus $k_H \equiv \frac{1}{k_L} < 1$

$$L = L_0 \mathrm{cn}\left(u_H; k_H\right)$$

$$\ell = \frac{\pi}{2} - \tan^{-1}\left[\frac{k_H\sqrt{1+n}\,\mathrm{sn}\left(u_H; k_H\right)}{\mathrm{dn}\left(u_H; k_H\right)}\right]$$

$$g = g_0 + cG_0\left(t - t_0\right) + \frac{a-c}{\sqrt{(a-b)(d-c)}}\mathrm{pn}\left(u_H; k_H^2 n, k_H\right)$$

Andoyer Angle (h, I, g, J, ℓ)

- 5 Angles Specifying Principal CS via Ang. Mom. Axis from Inertial CS
- Basis Matrix Expression of Principal CS

$$\begin{pmatrix}\mathbf{e}_A & \mathbf{e}_B & \mathbf{e}_C\end{pmatrix}^{\mathrm{T}} = \mathbf{R}_3(\ell)\mathbf{R}_1(J)\mathbf{R}_3(g)\mathbf{R}_1(I)\mathbf{R}_3(h)$$

- Relation with Generalized Momentum

$$p_\phi = G\cos J,\ p_\psi = G\cos I$$

$$p_\theta = G\sin J\sin(\phi-\ell) = G\sin I\sin(\psi - h)$$

Andoyer Angle (2)

- Basis Matrix in Inertial CS (Q: Derive)

$$\mathbf{e}_A = \begin{pmatrix} C_\ell C_g C_h - C_\ell S_g C_I S_h + S_\ell S_J S_I S_h - S_\ell C_J S_g C_h - S_\ell C_J C_g C_I S_h \\ C_\ell C_g S_h + C_\ell S_g C_I C_h - S_\ell S_J S_I C_h - S_\ell C_J S_g S_h + S_\ell C_J C_g C_I C_h \\ C_\ell S_g S_I + S_\ell S_J C_I \quad + S_\ell C_J C_g S_I \end{pmatrix}$$

$$\mathbf{e}_B = \begin{pmatrix} -S_\ell C_g C_h + S_\ell S_g C_I S_h + C_\ell S_J S_I S_h - C_\ell C_J S_g C_h - C_\ell C_J C_g C_I S_h \\ -S_\ell C_g S_h - S_\ell S_g C_I C_h - C_\ell S_J S_I C_h - C_\ell C_J S_g S_h + C_\ell C_J C_g C_I C_h \\ -S_\ell S_g S_I + C_\ell S_J C_I \quad + C_\ell C_J C_g S_I \end{pmatrix}$$

$$\mathbf{e}_C = \begin{pmatrix} C_J S_I S_h + S_J S_g C_h \quad + S_J C_g C_I S_h \\ -C_J S_I C_h + S_J S_g S_h \quad - S_J C_g C_I C_h \\ C_J C_I \quad - S_J C_g S_I \end{pmatrix}$$

Angular Momentum CS

- CS with Z-axis = Angular Momentum Axis $\mathbf{e}_G \equiv \dfrac{\mathbf{L}}{G}$
- Ang. Mom. CS Associated with Andoyer Angle

$$\begin{pmatrix}\mathbf{e}_E & \mathbf{e}_F & \mathbf{e}_G\end{pmatrix}^{\mathrm{T}} = \mathbf{R}_1(I)\,\mathbf{R}_3(h)$$

- Basis Expression in Andoyer Ang. Mom. CS

$$\mathbf{e}_X = \begin{pmatrix} C_h \\ -S_h C_I \\ S_h S_I \end{pmatrix}, \mathbf{e}_Y = \begin{pmatrix} S_h \\ C_h C_I \\ -C_h S_I \end{pmatrix}, \mathbf{e}_Z = \begin{pmatrix} 0 \\ S_I \\ C_I \end{pmatrix}$$

$$\mathbf{e}_A = \begin{pmatrix} C_\ell C_g - S_\ell C_J S_g \\ C_\ell S_g + S_\ell C_J C_g \\ S_\ell S_J \end{pmatrix}, \mathbf{e}_B = \begin{pmatrix} -S_\ell C_g - C_\ell C_J S_g \\ -S_\ell S_g + C_\ell C_J C_g \\ C_\ell S_J \end{pmatrix}, \mathbf{e}_C = \begin{pmatrix} S_J S_g \\ -S_J C_g \\ C_J \end{pmatrix}$$

Andoyer Angle (3)

- Angular Velocity Expression

$$\boldsymbol{\omega} = \frac{\mathrm{d}h}{\mathrm{dt}}\mathbf{e}_h + \frac{\mathrm{d}I}{\mathrm{dt}}\mathbf{e}_I + \frac{\mathrm{d}g}{\mathrm{dt}}\mathbf{e}_g + \frac{\mathrm{d}J}{\mathrm{dt}}\mathbf{e}_J + \frac{\mathrm{d}\ell}{\mathrm{dt}}\mathbf{e}_\ell$$

- Rotation Axis Vector in Ang. Mom. CS

$$\mathbf{e}_h = \mathbf{e}_Z = \mathbf{e}_F \sin h + \mathbf{e}_G \cos h$$

$$\mathbf{e}_I = \mathbf{e}_E, \mathbf{e}_g = \mathbf{e}_G, \mathbf{e}_J = \mathbf{e}_E \cos g + \mathbf{e}_F \sin g$$

$$\mathbf{e}_\ell = \mathbf{e}_C = \left(\mathbf{e}_E \sin g - \mathbf{e}_F \cos g\right) \sin J + \mathbf{e}_G \cos J$$

Angular Momentum

- Two Component Expressions of Ang. Mom.

$$\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = G \begin{pmatrix} \sin I \sin h \\ -\sin I \cos h \\ \cos I \end{pmatrix} = \begin{pmatrix} \sqrt{G^2 - H^2} \sin h \\ -\sqrt{G^2 - H^2} \cos h \\ H \end{pmatrix}$$

$$\begin{pmatrix} L_A \\ L_B \\ L_C \end{pmatrix} = \begin{pmatrix} A\omega_A \\ B\omega_B \\ C\omega_C \end{pmatrix} = G \begin{pmatrix} \sin J \sin \ell \\ \sin J \cos \ell \\ \cos J \end{pmatrix} = \begin{pmatrix} \sqrt{G^2 - L^2} \sin \ell \\ \sqrt{G^2 - L^2} \cos \ell \\ L \end{pmatrix}$$

Andoyer Variable

- Total Ang. Mom. + 5 Andoyer Angles

$$\left(G, h, I, g, J, \ell\right)$$

- Non-Canonical yet Geometrically Clear
  - Constant in Torque-Free Motion: G, h, I
  - Varying in Torque-Free Motion: g, J, ℓ

Andoyer Variable (2)

- Eq. of Motion of Small Varyinig Variable
  - Q: Derive from Canonical Eq. of Motion

$$\frac{\mathrm{d}G}{\mathrm{dt}} = \left(\frac{\partial V}{\partial g}\right)_{G,h,I,J,\ell}$$

$$\frac{\mathrm{d}h}{\mathrm{dt}} = \frac{1}{G\sin I}\left(\frac{\partial V}{\partial I}\right)_{G,h,g,J,\ell}$$

$$\frac{\mathrm{d}I}{\mathrm{dt}} = \frac{1}{G\sin I}\left[\cos I\left(\frac{\mathrm{d}G}{\mathrm{dt}}\right) - \left(\frac{\partial V}{\partial h}\right)_{G,I,g,J,\ell}\right]$$

Andoyer Variable (3)

- Eq. of Motion of Largely Varying Variables

$$\frac{\mathrm{d}g}{\mathrm{d}t}=G\left[\left(\frac{a+b}{2}\right)-\left(\frac{a-b}{2}\right)\cos 2\ell\right]-\left(\frac{\partial V}{\partial G}\right)_{L,H,\ell,g,h}$$

$$-\frac{1}{G}\left[\cot I\left(\frac{\partial V}{\partial I}\right)_{G,h,g,J,\ell}+\cot J\left(\frac{\partial V}{\partial J}\right)_{G,h,I,g,\ell}\right]$$

$$\frac{\mathrm{d}J}{\mathrm{d}t}=\left(\frac{a-b}{2}\right)G\sin J\sin 2\ell+\frac{1}{G\sin J}\left[\cos J\left(\frac{\mathrm{d}G}{\mathrm{d}t}\right)-\left(\frac{\partial V}{\partial \ell}\right)_{G,h,I,g,J}\right]$$

$$\frac{\mathrm{d}\ell}{\mathrm{d}t}=-G\cos J\left[\left(\frac{a+b}{2}-c\right)-\left(\frac{a-b}{2}\right)\cos 2\ell\right]+\frac{1}{G\sin J}\left(\frac{\partial V}{\partial J}\right)_{G,h,I,g,\ell}$$

Element of Free Motion

- Element ＝ 6 Constants of Motion
- Fukushima and Ishizaki (1994a,CMDA)
  - Andoyer Variables when J is Maximum

$$\left(G_0, h_0, I_0, g_0, J_0, t_0\right) \leftrightarrow \left(L_0, G_0, H_0, g_0, h_0, t_0\right)$$

- Constants

$$2T = aG_0^2 - \left(a-c\right)L_0^2 = G_0^2\left(a\sin^2 J_0 + c\cos^2 J_0\right)$$

$$d = a\sin^2 J_0 + c\cos^2 J_0$$

$$n = \frac{a-b}{a-c}$$

$$k = \sqrt{n}\tan J_0$$

Solution by Element

- Low Energy Case (Q: Derive)
  - a, b, c, n, σ are Const., Independent on Element

$$G = G_0, h = h_0, I = I_0$$

$$u \equiv G_0 \cos J_0 \sqrt{(a-c)(b-c)}(t-t_0)$$

$$g = g_0 + cG_0(t-t_0) + \frac{1}{\sigma \cos J_0} \mathrm{pn}(u; n, k)$$

$$\sigma = \frac{1}{\sqrt{1+n}}$$

$$J = \sin^{-1}\left[\sin J_0 \sqrt{1 + n\,\mathrm{sn}^2(u;k)}\right]$$

$$k = \sqrt{n} \tan J_0$$

$$n = \frac{a-b}{a-c}$$

$$\ell = \frac{\pi}{2} - \mathrm{am}(u;k) - \tan^{-1}\left[\frac{(1-\sigma)\,\mathrm{sn}(u;k)\,\mathrm{cn}(u;k)}{\sigma + (1-\sigma)\,\mathrm{sn}^2(u;k)}\right]$$

Orbit vs Rotation (2)

Item	Orbit	Rotation
Basic Sol.	Kepler Motion	Free Rotation
Classification	Ellipse, Hyperbola	Short-, Long-Axis
Key Param.	Eccentricity, e	Modulus, k
Special F.	Sol. of Kepler Eq.	Elliptic Function
Basic Angle	Ecc. Anomaly, E	Amplitude, am(u;k)
Can. Var.	Delauney Variable	Serret Variable
Element	Kepler Element	Free Rot. Element

Partial Derivative of V

- Complicated in Serret Var. (Cost of Simple T)

$$V = V(L, G, H, \ell, g, h) = V(G, h, I(G,H), g, J(G,L), \ell)$$

- Partial Derivative w.r.t. Action Variable

$$\left(\frac{\partial V}{\partial L}\right)_{G,H,\ell,g,h} = \left(\frac{\partial J}{\partial L}\right)_G \left(\frac{\partial V}{\partial J}\right)_{G,h,I,g,\ell} \qquad \left(\frac{\partial V}{\partial H}\right)_{L,G,\ell,g,h} = \left(\frac{\partial I}{\partial H}\right)_G \left(\frac{\partial V}{\partial I}\right)_{G,h,g,J,\ell}$$

$$\left(\frac{\partial V}{\partial G}\right)_{L,H,\ell,g,h} = \left(\frac{\partial V}{\partial G}\right)_{h,I,g,J,\ell} + \left(\frac{\partial J}{\partial G}\right)_L \left(\frac{\partial V}{\partial J}\right)_{G,h,I,g,\ell} + \left(\frac{\partial I}{\partial G}\right)_H \left(\frac{\partial V}{\partial I}\right)_{G,h,g,J,\ell}$$

- Partial Derivative of Auxiliary Angle

$$\left(\frac{\partial J}{\partial L}\right)_G = \frac{-1}{G\sin J}, \left(\frac{\partial J}{\partial G}\right)_L = \frac{\cos J}{G\sin J} \qquad \left(\frac{\partial I}{\partial H}\right)_G = \frac{-1}{G\sin I}, \left(\frac{\partial I}{\partial G}\right)_H = \frac{\cos I}{G\sin I}$$

Angle PD and Torque (2)

- PD of Andoyer and Euler Angle

$$\left(\frac{\partial V}{\partial J}\right)_{\ell,g,I,h} = \left(\frac{\partial i}{\partial J}\right)_{g,I}\left(\frac{\partial V}{\partial \psi}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial J}\right)_{g,I}\left(\frac{\partial V}{\partial \theta}\right)_{\psi,\phi} + \left(\frac{\partial j}{\partial J}\right)_{g,I}\left(\frac{\partial V}{\partial \phi}\right)_{\psi,\theta}$$

$$\left(\frac{\partial V}{\partial I}\right)_{\ell,J,g,h} = \left(\frac{\partial i}{\partial I}\right)_{J,g}\left(\frac{\partial V}{\partial \psi}\right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial I}\right)_{J,g}\left(\frac{\partial V}{\partial \theta}\right)_{\psi,\phi} + \left(\frac{\partial j}{\partial I}\right)_{J,g}\left(\frac{\partial V}{\partial \phi}\right)_{\psi,\theta}$$

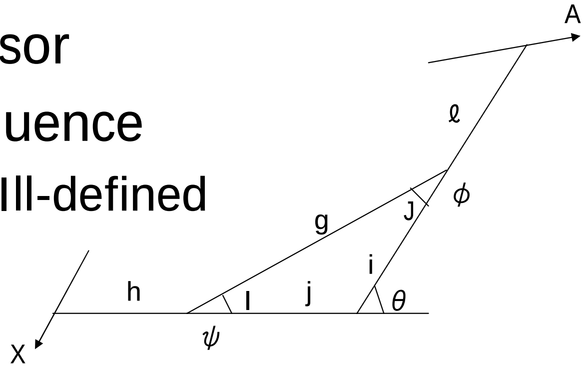
- PD w.r.t. Andoyer Angle I

$$\left(\frac{\partial i}{\partial I}\right)_{J,g} = \frac{-\sin i \cos i}{\tan I} \quad \left(\frac{\partial j}{\partial I}\right)_{J,g} = \frac{\sin g \cos^2 j}{\sin J \cos^2 I} \quad \left(\frac{\partial \theta}{\partial I}\right)_{J,g} = \frac{\sin \theta}{\sin I \sin J + \cos I \cos g \sin J}$$

- PD w.r.t. J: by Symmetric Change I,i⇔J,j

Small Angle Difficulty

- Common to Serret and Andoyer Variable
- sin I, sin J in Divisor
- Cause: 3-1-3 Sequence
  - Ex.: h and g are Ill-defined when I~0



- Recipe
  - 1-2-3 Sequence + Appropriate CS

Canonical Transf. assoc. with Spherical Rectangular Triangle

- Fukushima (1994,CMDA)

$$\left(X,Y;x,y\right)\rightarrow\left(U,V;u,v\right)$$

- Transformation (Q: Show Canonicity)

$$U=X$$

$$V=\sqrt{X^2-Y^2}\sin y$$

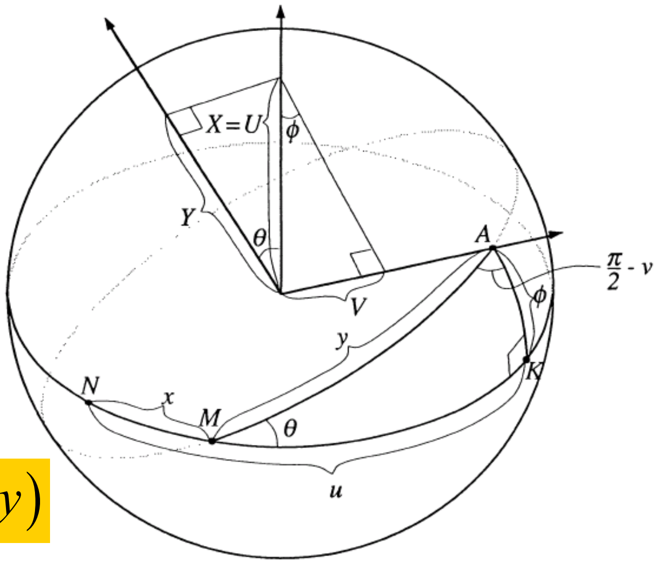
$$u=x+\tan^{-1}\left(\frac{Y}{X}\tan y\right)$$

$$v=\tan^{-1}\left(\frac{\sqrt{X^2-Y^2}}{Y}\cos y\right)$$

Canonical Transf. (2)

- Aux. Angle

$$\theta \equiv \sin^{-1}\left(\frac{Y}{X}\right)$$



$$\phi \equiv \sin^{-1}\left(\sin\theta \sin y\right)$$

Canonical Transf. (3)

- Matrix Relation of Angle Variable

$$\mathbf{R}_3(x)\mathbf{R}_1(\theta) = \mathbf{R}_1(u)\mathbf{R}_2(-\phi)\mathbf{R}_3(v)$$

- Transf. Formula (Q: Confirm)

$$u = \tan^{-1}(\tan\theta\cos y)$$

$$\tau_\theta \equiv \tan\frac{\theta}{2}$$

$$v = \tan^{-1}(\cos\theta\tan y) = y - \tan^{-1}\left(\frac{\tau_\theta^2 \sin 2y}{1+\tau_\theta^2 \cos 2y}\right)$$

New Canonical Var.

- Double Appl. of Canonical Transformation
  $$(L, G, H; \ell, g, h) \to (X, F, W; x, f, w)$$
- New Action Variable: $X \equiv L_A, F \equiv G, W \equiv L_X$
- New Aux. Angle: u, z, $f \equiv u + g + z$
- New Form of Kinetic Energy
  $$2T = cF^2 + (a-c)X^2 + (b-c)(F^2 - X^2)\sin^2 x$$
  - After Average w.r.t. x → Elliptic Form

New Action Variable

- Expression of New Action Variable in terms of Serret Canonical Variable

$$X \equiv L_A = \sqrt{G^2 - L^2}\sin\ell = G\sin J\sin\ell$$

$$F \equiv G$$

$$W \equiv L_X = \sqrt{G^2 - H^2}\sin h = G\sin I\sin h$$

New Aux. Angle

- New Aux. Angle: y, z, u, and v

$$x = \tan^{-1}\left(\tan J \cos \ell\right)$$
$$y = \sin^{-1}\left(\sin J \sin \ell\right)$$
$$z = \tan^{-1}\left(\cos J \tan \ell\right)$$

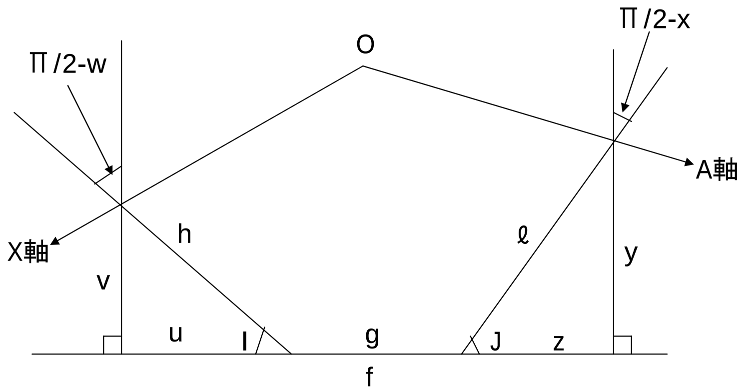
$$w = \tan^{-1}\left(\tan I \cos h\right)$$
$$v = \sin^{-1}\left(\sin I \sin h\right)$$
$$u = \tan^{-1}\left(\cos I \tan h\right)$$

$$f \equiv u + g + z$$

New Angle V. (w, v, f, y, x)

- Basis Matrix (Note Signature of y)

$$\left(\mathbf{e}_A \quad \mathbf{e}_B \quad \mathbf{e}_C\right)^{\mathrm{T}} = \mathbf{R}_1(x)\mathbf{R}_2(-y)\mathbf{R}_3(f)\mathbf{R}_2(v)\mathbf{R}_1(w)$$



New Angle Var. (2)

- Two Expression of Ang. Mom. Vector

$$\begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix} = F \begin{pmatrix} \sin v \\ -\cos v \sin w \\ \cos v \cos w \end{pmatrix} = \begin{pmatrix} W \\ -\sqrt{F^2 - W^2} \sin w \\ \sqrt{F^2 - W^2} \cos w \end{pmatrix}$$

$$\begin{pmatrix} L_A \\ L_B \\ L_C \end{pmatrix} = \begin{pmatrix} A\omega_A \\ B\omega_B \\ C\omega_C \end{pmatrix} = F \begin{pmatrix} \sin y \\ \cos y \sin x \\ \cos y \cos x \end{pmatrix} = \begin{pmatrix} X \\ \sqrt{F^2 - X^2} \sin x \\ \sqrt{F^2 - X^2} \cos x \end{pmatrix}$$

Ang. Mom. CS (2)

- Ang. Mom. CS for New Angle Var.

$$\mathbf{e}_R \equiv \frac{\mathbf{L}}{F}$$

$$\begin{pmatrix}\mathbf{e}_P & \mathbf{e}_Q & \mathbf{e}_R\end{pmatrix}^{\mathrm{T}} = \mathbf{R}_2(v)\mathbf{R}_1(w)$$

- Basis Expr. in New Ang. Mom. CS

$$\mathbf{e}_X = \begin{pmatrix} C_v \\ 0 \\ S_v \end{pmatrix}, \mathbf{e}_Y = \begin{pmatrix} S_v S_w \\ C_w \\ -C_v S_w \end{pmatrix}, \mathbf{e}_Z = \begin{pmatrix} -S_v C_w \\ S_w \\ C_v C_w \end{pmatrix}$$

$$\mathbf{e}_A = \begin{pmatrix} C_y C_f \\ C_y S_f \\ S_f \end{pmatrix}, \mathbf{e}_B = \begin{pmatrix} -C_x S_f - S_x S_y C_f \\ C_x C_f - S_x S_y S_f \\ S_x C_y \end{pmatrix}, \mathbf{e}_C = \begin{pmatrix} S_x S_f - C_x S_y C_f \\ -S_x C_f - C_x S_y S_f \\ C_x C_y \end{pmatrix}$$

New Angle Var. (3)

- Angular Velocity (Again Note Signature)

$$\boldsymbol{\omega} = \frac{\mathrm{d}w}{\mathrm{d}t}\mathbf{e}_w + \frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{e}_v + \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{e}_f - \frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{e}_y + \frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{e}_x$$

- Rotation Axis Vector in New Ang. Mom. CS

$$\mathbf{e}_w = \mathbf{e}_X = \mathbf{e}_P \cos v + \mathbf{e}_R \sin v$$

$$\mathbf{e}_v = \mathbf{e}_Q, \mathbf{e}_f = \mathbf{e}_R, \mathbf{e}_y = -\mathbf{e}_P \sin f + \mathbf{e}_Q \cos f$$

$$\mathbf{e}_x = \mathbf{e}_A = \left(\mathbf{e}_P \cos f + \mathbf{e}_Q \sin f\right)\cos y + \mathbf{e}_R \sin y$$

Can. Eq. of Motion (2)

- Eq. of Motion for New Canonical Variable

$$(X,F,W;x,f,w)$$

$$\frac{dF}{dt} = \left(\frac{\partial V}{\partial f}\right)_{X,F,W,x,w} \qquad \frac{dX}{dt} = -\left(\frac{b-c}{2}\right)\left(F^2 - X^2\right)\sin 2x + \left(\frac{\partial V}{\partial x}\right)_{X,F,W,f,w}$$

$$\frac{dW}{dt} = \left(\frac{\partial V}{\partial w}\right)_{X,F,W,x,f} \qquad \frac{dx}{dt} = X\left[(a-c)-(b-c)\sin^2 x\right] - \left(\frac{\partial V}{\partial X}\right)_{F,W,x,f,w}$$

$$\frac{dw}{dt} = -\left(\frac{\partial V}{\partial W}\right)_{X,F,x,f,w} \qquad \frac{df}{dt} = F\left[c+(b-c)\sin^2 x\right] - \left(\frac{\partial V}{\partial F}\right)_{X,W,x,f,w}$$

New Variable

- Total Ang. Mom. + 5 New Angle $\left(F, w, v, f, y, x\right)$
- EoM of Small Varying Var.: F, w, and v

$$\frac{\mathrm{d}F}{\mathrm{d}t}=\left(\frac{\partial V}{\partial f}\right)_{X,F,W,x,w}, \quad \frac{\mathrm{d}w}{\mathrm{d}t}=-\left(\frac{\partial V}{\partial W}\right)_{X,F,x,f,w}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\left(\frac{\partial V}{\partial w}\right)_{X,F,W,x,f}-\left(\frac{\tan v}{F}\right)\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)$$

New Variable (2)

- Eq. of Motion of Largely Varying Variables: f, y, and x

$$\frac{\mathrm{d}f}{\mathrm{d}t} = F\left[\left(\frac{b+c}{2}\right) - \left(\frac{b-c}{2}\right)\cos 2x\right] - \left(\frac{\partial V}{\partial F}\right)_{X,W,x,f,w}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\left(\frac{b-c}{2}\right) F\cos y \sin 2x + \frac{1}{F\cos y}\left[\left(\frac{\partial V}{\partial x}\right)_{X,F,W,f,w} - \sin y\left(\frac{\mathrm{d}F}{\mathrm{d}t}\right)\right]$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F\left[\left(\frac{2a-b-c}{2}\right) + \left(\frac{b-c}{2}\right)\cos 2x\right]\sin y - \left(\frac{\partial V}{\partial X}\right)_{F,W,x,f,w}$$

Torque-Free Solution (4)

- V=0 → Trivial Solution $F = F_0, W = W_0, w = w_0$
- Basic Equation

$$\frac{\mathrm{d}X}{\mathrm{dt}} = -\left(\frac{b-c}{2}\right)\left(F_0^2 - X^2\right)\sin 2x$$

$$\frac{\mathrm{d}x}{\mathrm{dt}} = X\left[\left(\frac{2a-b-c}{2}\right)+\left(\frac{b-c}{2}\right)\cos 2x\right]$$

$$\frac{\mathrm{d}f}{\mathrm{dt}} = F_0\left[\left(\frac{b+c}{2}\right)-\left(\frac{b-c}{2}\right)\cos 2x\right]$$

Torque-Free Solution (5)

- Case B=C: Simple
  - X, F, W, and w: Constant

    $$X = X_0,\ F = F_0,\ W = W_0,\ w = w_0$$

  - x and f: Linear Function of Time

    $$x = x_0 + n_x\left(t - t_0\right),\ f = f_0 + n_f\left(t - t_0\right)$$

  - x: Slow Motion, f: Fast Motion

    $$n_x = \left(a - c\right)X_0 > 0,\ n_f = cF_0 > 0$$

Torque-Free Solution (6)

- Case A=B<C: by Trigonometric Function
  - Energy Integral

$$2T_0 \equiv F_0^2 d = aX^2 + \left(a\sin^2 x + c\cos^2 x\right)\left(F_0^2 - X^2\right)$$

  - Deleting x

$$\frac{\mathrm{d}X}{\mathrm{dt}} = -F_0\sqrt{\left(a-d\right)\left(a-c\right)\left(X_0^2 - X^2\right)} \qquad X_0 \equiv F_0\sqrt{\frac{d-c}{a-c}}$$

  - Solution of X (Q: Obtain Solution of x and f)

$$X = X_0 \cos\varphi \qquad \varphi \equiv F_0\sqrt{\left(a-d\right)\left(d-c\right)}\left(t-t_0\right)$$

Equal Energy Curve (3)

- Free Rot. → F=Const. → Again Freedom 2
- New Normalized Kin. Energy

$$K \equiv \frac{4T - 2aF^2}{(2a-b-c)F^2} = -\cos^2 y (1 + P\cos 2x)$$

$$P \equiv \frac{b-c}{2a-b-c}$$

$$0 \le P \le 1$$

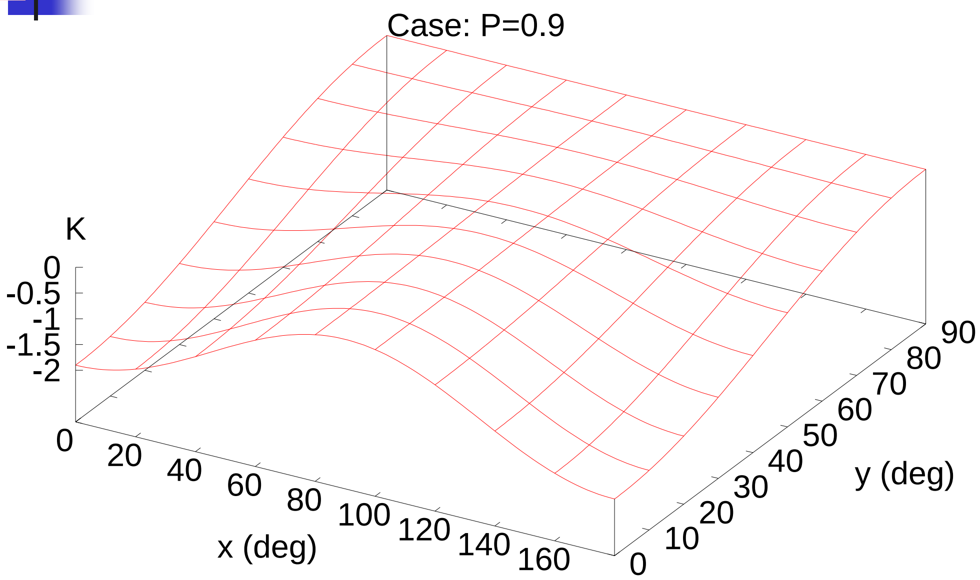
$$-(1+P) \le K \le 0$$

- New Equal Energy Curve
  - Libration (K>-1+P) vs Rotation (K<-1+P)
  - A~B → P~1 → Libration is Major

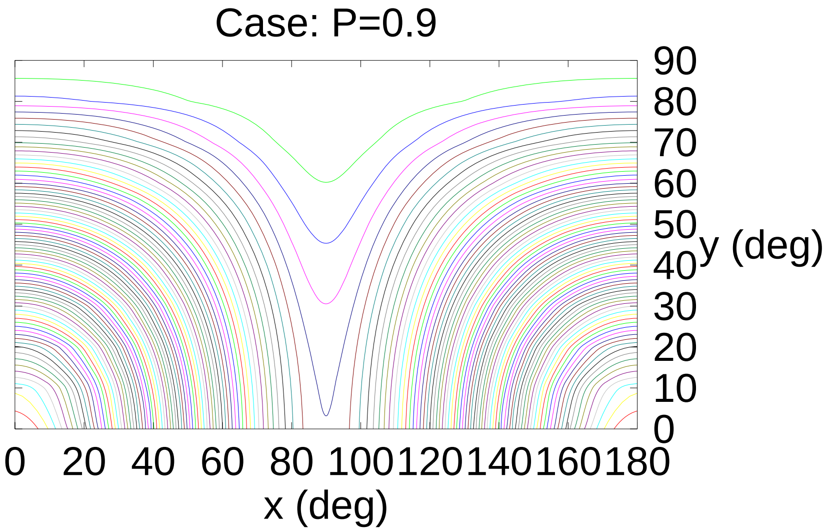
$$1 - P_{\mathrm{Earth}} = 1.337 \times 10^{-3}$$

$$P_{\mathrm{Moon}} = 0.539$$

Normalized Energy (2)



Equal Energy Curve (4)



Torque-Free Solution (7)

- Case B<C: Approx. Librating Sol. Exists
  - Ignore 2nd and Higher Order of X and x

$$\frac{\mathrm{d}X}{\mathrm{dt}} = -(b-c)F_0^2 x, \quad \frac{\mathrm{d}x}{\mathrm{dt}} = (a-c)X, \quad \frac{\mathrm{d}f}{\mathrm{dt}} = cF_0$$

  - X and x: Slow Libration, f: Fast Linear Motion

$$X = X_0 \cos n_X t - X_1 \sin n_X t, \quad x = x_0 \cos n_X t + x_1 \sin n_X t, \quad f = f_0 + n_f t$$

$$X_1 \equiv \frac{(b-c)F_0^2 x_0}{n_X}, \quad x_1 \equiv \frac{(a-c)X_0}{n_X} \qquad n_X = F_0\sqrt{(a-c)(b-c)} \qquad n_f = cF_0$$

Torque-Free Solution (8)

- Case A<B<C
  - Energy Integral

$$2T_0 \equiv F_0^2 d = aX^2 + \left(b\sin^2 x + c\cos^2 x\right)\left(F_0^2 - X^2\right)$$

  - Deleting x

$$\frac{\mathrm{d}X}{\mathrm{dt}} = -\sqrt{(a-b)(a-c)\left(X_0^2 - X^2\right)\left(X^2 + S_1\right)}$$

$$X_0 \equiv F_0\sqrt{\frac{d-c}{a-c}} \qquad S_1 \equiv \left(\frac{b-d}{a-b}\right)F_0^2$$

Torque-Free Solution (9)

- Low Energy (d<b) Case Solution
  - Fukushima (1994, CMDA)

$$\frac{k_L^2}{n} = \frac{d-c}{a-d}$$

$$X = X_0 \mathrm{cn}\left(u_L; k_L\right)$$

$$x = \tan^{-1}\left[\left(\frac{k_L}{\sqrt{n}}\right)\frac{\sqrt{1+n\,\mathrm{sn}\left(u_L; k_L\right)}}{\mathrm{dn}\left(u_L; k_L\right)}\right]$$

$$f = f_0 + aG_0\left(t - t_0\right) + \frac{a-c}{\sqrt{\left(a-d\right)\left(b-c\right)}}\mathrm{pn}\left(u_L; \frac{k_L^2}{n}, k_L\right)$$

- Q: Obtain Other Case Solution

8. Numerical Integration of Rotation

- Various Sets of Variable
  - Euler: Euler Angle and Ang. Vel. $\psi,\theta,\phi,\omega_A,\omega_B,\omega_C$
  - Lagrange: Euler Angle and Time Deriv. $\psi,\theta,\phi,\dot{\psi},\dot{\theta},\dot{\phi}$
  - Ang. Mom.: Euler Angle and AM $\psi,\theta,\phi,L_A,L_B,L_C$
  - Andoyer: Andoyer Variable G,h,I,g,J,ℓ
  - Inertial: Euler Angle and AM $\psi,\theta,\phi,L_X,L_Y,L_Z$
  - Element: G,h,I,τ,u,s
- Small Angle Difficulty $\theta \sim 0$ $I \sim 0$ $J \sim 0$
- Encke's Method and Fast Rotation Difficulty

Euler Set

- Variable $(\psi,\theta,\phi,\omega_A,\omega_B,\omega_C)$

$$\frac{\mathrm{d}\psi}{\mathrm{dt}}=\frac{\omega_A\sin\phi+\omega_B\cos\phi}{\sin\theta}$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}}=\omega_A\cos\phi-\omega_B\sin\phi$$

$$\frac{\mathrm{d}\phi}{\mathrm{dt}}=\omega_C-\left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)\cos\theta$$

$$\frac{\mathrm{d}\omega_A}{\mathrm{dt}}=W_A$$

$$\frac{\mathrm{d}\omega_B}{\mathrm{dt}}=W_B$$

$$\frac{\mathrm{d}\omega_C}{\mathrm{dt}}=W_C$$

$$W_A\equiv aN_A-\alpha\omega_B\omega_C,W_B\equiv bN_B-\beta\omega_C\omega_A,W_C\equiv cN_C-\gamma\omega_A\omega_B$$

Lagrange Set

- Variable $\left(\psi,\theta,\phi,\dot{\psi},\dot{\theta},\dot{\phi}\right)$
- Eq. of Motion (Q: Confirm)

$$\frac{d^2\psi}{dt^2}=\frac{1}{\sin\theta}\left[W_A\sin\phi+W_B\cos\phi+\left(\frac{d\theta}{dt}\right)\left\{\left(\frac{d\phi}{dt}\right)-\left(\frac{d\psi}{dt}\right)\cos\theta\right\}\right]$$

$$\frac{d^2\theta}{dt^2}=W_A\cos\phi-W_B\sin\phi-\left(\frac{d\phi}{dt}\right)\left[\left(\frac{d\psi}{dt}\right)\sin\theta\right]$$

$$\frac{d^2\phi}{dt^2}=W_C-\left(\frac{d^2\psi}{dt^2}\right)\cos\theta+\left(\frac{d\theta}{dt}\right)\left[\left(\frac{d\psi}{dt}\right)\sin\theta\right]$$

Angular Momentum Set

- Variable $(\psi, \theta, \phi, L_A, L_B, L_C)$

$$\frac{d\psi}{dt} = \frac{aL_A \sin\phi + bL_B \cos\phi}{\sin\theta}$$

$$\frac{d\theta}{dt} = aL_A \cos\phi - bL_B \sin\phi$$

$$\frac{d\phi}{dt} = cL_C - \left(\frac{d\psi}{dt}\right)\cos\theta$$

$$\frac{dL_A}{dt} = N_A - (b-c)L_C L_B$$

$$\frac{dL_B}{dt} = N_B - (c-a)L_A L_C$$

$$\frac{dL_C}{dt} = N_C - (a-b)L_B L_A$$

Defect of Existing Sets

- Case of Free Rotation (or A=B)
  - Reality: Complexity 2 (or 4)
    - Complexity=Number of Essential Freedom
    - Euler Set, Ang. Mom. Set: Complexity 6 (or 5)
    - Lagrange Set: Always Complexity 6
- Andoyer Set
  - Complexity 2 (Free Rot.), or 6 (A=B)
- Inertial Set
  - Complexity 3 (Free Rot.), or 4 (A=B)

Andoyer Set

- Andoyer Variable (G, h, I, g, J, ℓ)
- Ang. Mom. CS $(\mathbf{e}_E \quad \mathbf{e}_F \quad \mathbf{e}_G)$
  - G, h, I: Use of Torque $\mathbf{N} = N_A\mathbf{e}_A + N_B\mathbf{e}_B + N_C\mathbf{e}_C$

$$\frac{\mathrm{d}G}{\mathrm{dt}} = N_G, \frac{\mathrm{d}h}{\mathrm{dt}} = \frac{N_E}{G\sin I}, \frac{\mathrm{d}I}{\mathrm{dt}} = -\left(\frac{N_F}{G}\right) \qquad N_E \equiv \mathbf{N}\cdot\mathbf{e}_E, \text{ etc.}$$

  - g, J, ℓ: Inverse Relation of Angular Velocity

$$\frac{\mathrm{d}g}{\mathrm{dt}}\mathbf{e}_G + \frac{\mathrm{d}J}{\mathrm{dt}}\mathbf{e}_J + \frac{\mathrm{d}\ell}{\mathrm{dt}}\mathbf{e}_C = \boldsymbol{\omega} - \boldsymbol{\Omega} \qquad \boldsymbol{\Omega} \equiv \frac{\mathrm{d}h}{\mathrm{dt}}\mathbf{e}_Z + \frac{\mathrm{d}I}{\mathrm{dt}}\mathbf{e}_E$$

Andoyer Set (2)

- Rotation of Ang. Mom. CS w.r.t. Inertial CS

$$\boldsymbol{\Omega} \equiv \frac{\mathrm{d}h}{\mathrm{d}t}\mathbf{e}_Z + \frac{\mathrm{d}I}{\mathrm{d}t}\mathbf{e}_E \qquad \mathbf{e}_Z = \mathbf{e}_F \sin I + \mathbf{e}_G \cos I$$

- Angular Velocity Component of CS Rotation

$$\Omega_E = -\left(\frac{N_F}{G}\right), \quad \Omega_F = \frac{N_E}{G}, \quad \Omega_G = \frac{N_E}{G}\cot I$$

$$\Omega_J = \Omega_E \cos g + \Omega_F \sin g, \quad \Omega_M = -\Omega_E \sin g + \Omega_F \cos g$$
$$\Omega_K = \Omega_G \sin J + \Omega_M \cos J, \quad \Omega_C = \Omega_G \cos J - \Omega_M \sin J$$

Andoyer Set (3)

- Eq. of Motion of g, J, and ℓ (Q: Show)

$$\frac{\mathrm{d}g}{\mathrm{dt}} = G\left[\left(\frac{a+b}{2}\right)-\left(\frac{a-b}{2}\right)\cos 2\ell\right]-\left(\Omega_G+\Omega_M\cot J\right)$$

$$\frac{\mathrm{d}J}{\mathrm{dt}} = \left(\frac{a-b}{2}\right)G\sin J\sin 2\ell-\left[\Omega_J-\left(\frac{N_G}{G}\right)\cot J\right]$$

$$\frac{\mathrm{d}\ell}{\mathrm{dt}} = -G\cos J\left[\left(\frac{a+b}{2}-c\right)-\left(\frac{a-b}{2}\right)\cos 2\ell\right]-\frac{\Omega_M}{\sin J}$$

Inertial Set

- Variables $(\psi, \theta, \phi, \mathbf{L})$
- Eq. of Motion for Angular Momentum $\frac{d\mathbf{L}}{dt} = \mathbf{N}$
  - Complexity 3 in Free Rotation $\mathbf{N} = N_A \mathbf{e}_A + N_B \mathbf{e}_B + N_C \mathbf{e}_C$
- Introduction of Intermediate Coord. System

$$(\mathbf{e}_S \quad \mathbf{e}_T \quad \mathbf{e}_C)^{\mathrm{T}} = \mathbf{R}_1(\theta)\mathbf{R}_3(\psi)$$

$$\mathbf{e}_A = \mathbf{e}_S \cos\phi + \mathbf{e}_T \sin\phi$$
$$\mathbf{e}_B = -\mathbf{e}_S \sin\phi + \mathbf{e}_T \cos\phi$$

$$\mathbf{e}_S = \begin{pmatrix} \cos\psi \\ \sin\psi \\ 0 \end{pmatrix} \quad \mathbf{e}_T = \begin{pmatrix} -\cos\theta\sin\psi \\ \cos\theta\cos\psi \\ \sin\theta \end{pmatrix} \quad \mathbf{e}_C = \begin{pmatrix} \sin\theta\sin\psi \\ -\sin\theta\cos\psi \\ \cos\theta \end{pmatrix}$$

Inertial Set (2)

- Eq. of Motion for Euler Angle (Q: Derive)

$$\frac{\mathrm{d}\psi}{\mathrm{dt}} = \frac{1}{\sin\theta}\left[\left(\frac{a+b}{2}\right)L_T + \left(\frac{a-b}{2}\right)\left(L_S \sin 2\phi - L_T \cos 2\phi\right)\right]$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}} = \left(\frac{a+b}{2}\right)L_S + \left(\frac{a-b}{2}\right)\left(L_S \cos 2\phi + L_T \sin 2\phi\right)$$

$$\frac{\mathrm{d}\phi}{\mathrm{dt}} = cL_C - \left(\frac{\mathrm{d}\psi}{\mathrm{dt}}\right)\cos\theta$$

$$L_S = L_X \cos\psi + L_Y \sin\psi, L_T = -\left(L_X \sin\psi - L_Y \cos\psi\right)\cos\theta + L_Z \sin\theta$$

$$L_C = \left(L_X \sin\psi - L_Y \cos\psi\right)\sin\theta + L_Z \cos\theta$$

Inertial Set (3)

- Case A=B
  - Integral $N_C = 0 \rightarrow L_C = \text{const.}$
  - Reducing Number of Variable

$$L_Z = \frac{L_C - (L_X \sin\psi - L_Y \cos\psi)\sin\theta}{\cos\theta}$$

  - Note 1: Rot. Angle is Not Needed in Torque Eval.
  - Note 2: Rot. Angle is Not Needed in Ang. Vel. Eval.
  - Then, Complexity 4 (ψ, θ, L_X, L_Y)

Inertial Set (4)

- Case A=B (2)
  - Eq. of Motion for Prec. and Incl. Angle

$$\frac{\mathrm{d}\psi}{\mathrm{dt}} = \frac{aL_T}{\sin\theta}, \qquad \frac{\mathrm{d}\theta}{\mathrm{dt}} = aL_S$$

  - Rot. Angle Integration Can Be Done Later

$$\frac{\mathrm{d}\phi}{\mathrm{dt}} = cL_C - aL_T \cot\theta$$

Inertial Set (5)

- Case A=B (3)
  - Eq. of Motion for Angular Momentum

$$\frac{\mathrm{d}L_X}{\mathrm{dt}} = N_S \cos\psi - N_T \cos\theta \sin\psi$$

$$\frac{\mathrm{d}L_Y}{\mathrm{dt}} = N_S \sin\psi + N_T \cos\theta \cos\psi$$

$$r_S = \mathbf{r} \cdot \mathbf{e}_S, \mathrm{etc.}$$

$$N_S = -(C-A)\sum \frac{3\mu}{r^5} r_C r_T, N_T = +(C-A)\sum \frac{3\mu}{r^5} r_C r_S$$

Small Angle Difficulty

- Destiny of 3-1-3 Convention
- Apparent Difficulty: θ ~0, I~0
  - Solution: Choosing Suitable CS
  - Ex.: Ecliptic CS for Earth Rotation
- True Difficulty: J~0
  - Essential, Independent on CS Adopted
  - Solution: Another Convention (Ex. 1-2-3)

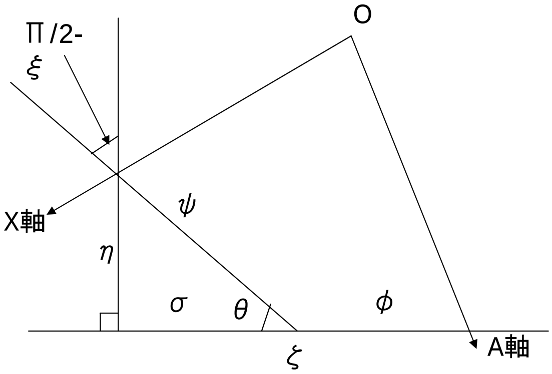
1-2-3 Convention

- Matrix Relation

$$\mathbf{R}_3(\phi)\mathbf{R}_1(\theta)\mathbf{R}_3(\psi)$$
$$=\mathbf{R}_3(\zeta)\mathbf{R}_2(\eta)\mathbf{R}_1(\xi)$$

- Angle Transf.

$$\xi=\tan^{-1}\left(\tan\theta\cos\psi\right)$$
$$\eta=\sin^{-1}\left(\sin\theta\sin\psi\right)$$
$$\zeta=\phi+\sigma=\phi+\psi-\tan^{-1}\left(\frac{\tau_\theta^2\sin 2\psi}{1+\tau_\theta^2\cos 2\psi}\right)$$
$$\tau_\theta=\tan\frac{\theta}{2}$$



1-2-3 Euler Angle

- Angular Velocity

$$\boldsymbol{\omega} = \frac{\mathrm{d}\xi}{\mathrm{dt}}\mathbf{e}_{\xi} + \frac{\mathrm{d}\eta}{\mathrm{dt}}\mathbf{e}_{\eta} + \frac{\mathrm{d}\zeta}{\mathrm{dt}}\mathbf{e}_{\zeta}$$

- Rotation Axis Vector

$$\mathbf{e}_{\xi} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_{\eta} = \begin{pmatrix} 0 \\ \cos\xi \\ \sin\xi \end{pmatrix}, \mathbf{e}_{\zeta} = \mathbf{e}_{C} = \begin{pmatrix} \sin\eta \\ -\cos\eta\sin\xi \\ \cos\eta\cos\xi \end{pmatrix}$$

- Basis of Principal Coordinate System

$$\mathbf{e}_{A} = \begin{pmatrix} \cos\zeta\cos\eta \\ \cos\zeta\sin\eta\sin\xi + \sin\zeta\cos\xi \\ -\cos\zeta\sin\eta\cos\xi + \sin\zeta\sin\xi \end{pmatrix}, \mathbf{e}_{B} = \begin{pmatrix} -\sin\zeta\cos\eta \\ -\sin\zeta\sin\eta\sin\xi + \cos\zeta\cos\xi \\ \sin\zeta\sin\eta\cos\xi + \cos\zeta\sin\xi \end{pmatrix}$$

Angular Velocity

- Inertial CS
  - OK for $\eta \sim 0$

$$\omega_X = \dot{\xi} + \dot{\zeta}\sin\eta$$
$$\omega_Y = \dot{\eta}\cos\xi - \dot{\zeta}\cos\eta\sin\xi$$
$$\omega_Z = \dot{\eta}\sin\xi + \dot{\zeta}\cos\eta\cos\xi$$

$$\frac{d\xi}{dt} = \omega_X - \left(\frac{d\zeta}{dt}\right)\sin\eta$$
$$\frac{d\eta}{dt} = \omega_Y\cos\xi + \omega_Z\sin\xi$$
$$\frac{d\zeta}{dt} = \frac{-\omega_Y\sin\xi + \omega_Z\cos\xi}{\cos\eta}$$

- Replacing Angle Component in Eq. of Motion for Inertial Set

Angular Velocity (2)

- Principal CS
  - OK for $\eta \sim 0$

$$\omega_A = \dot{\xi}\cos\zeta\cos\eta + \dot{\eta}\sin\zeta$$
$$\omega_B = -\dot{\xi}\sin\zeta\cos\eta + \dot{\eta}\cos\zeta$$
$$\omega_C = \dot{\xi}\sin\eta + \dot{\zeta}$$

$$\frac{\mathrm{d}\xi}{\mathrm{dt}} = \frac{\omega_A\cos\zeta - \omega_B\sin\zeta}{\cos\eta}$$
$$\frac{\mathrm{d}\eta}{\mathrm{dt}} = \omega_A\sin\zeta + \omega_B\cos\zeta$$
$$\frac{\mathrm{d}\zeta}{\mathrm{dt}} = \omega_C - \left(\frac{\mathrm{d}\xi}{\mathrm{dt}}\right)\sin\eta$$

- Replacing Angle Component in Eq. of Motion for Euler Set

Euler Set (2)

- Variable $(\xi, \eta, \zeta, \omega_A, \omega_B, \omega_C)$
- Eq. of Motion

$$\frac{\mathrm{d}\xi}{\mathrm{dt}} = \frac{\omega_A \cos\zeta - \omega_B \sin\zeta}{\cos\eta}$$

$$\frac{\mathrm{d}\eta}{\mathrm{dt}} = \omega_A \sin\zeta + \omega_B \cos\zeta$$

$$\frac{\mathrm{d}\zeta}{\mathrm{dt}} = \omega_C - \left(\frac{\mathrm{d}\xi}{\mathrm{dt}}\right)\sin\eta$$

$$\frac{\mathrm{d}\omega_A}{\mathrm{dt}} = W_A$$

$$\frac{\mathrm{d}\omega_B}{\mathrm{dt}} = W_B$$

$$\frac{\mathrm{d}\omega_C}{\mathrm{dt}} = W_C$$

Lagrange Set (2)

- Variable $\left(\xi,\eta,\zeta,\dot{\xi},\dot{\eta},\dot{\zeta}\right)$
- Eq. of Motion (Q: Confirm)

$$\frac{\mathrm{d}^2\xi}{\mathrm{dt}^2}=\frac{1}{\cos\eta}\left[W_A\cos\zeta-W_B\sin\zeta-\left(\frac{\mathrm{d}\eta}{\mathrm{dt}}\right)\left\{\left(\frac{\mathrm{d}\zeta}{\mathrm{dt}}\right)-\left(\frac{\mathrm{d}\xi}{\mathrm{dt}}\right)\sin\eta\right\}\right]$$

$$\frac{\mathrm{d}^2\eta}{\mathrm{dt}^2}=W_A\sin\zeta+W_B\cos\zeta+\left(\frac{\mathrm{d}\zeta}{\mathrm{dt}}\right)\left[\left(\frac{\mathrm{d}\xi}{\mathrm{dt}}\right)\cos\eta\right]$$

$$\frac{\mathrm{d}^2\zeta}{\mathrm{dt}^2}=W_C-\left(\frac{\mathrm{d}^2\xi}{\mathrm{dt}^2}\right)\sin\eta-\left(\frac{\mathrm{d}\eta}{\mathrm{dt}}\right)\left[\left(\frac{\mathrm{d}\xi}{\mathrm{dt}}\right)\cos\eta\right]$$

Angular Mom. Set (2)

- Variable $(\xi,\eta,\zeta,L_A,L_B,L_C)$
- Eq. of Motion

$$\frac{\mathrm{d}\xi}{\mathrm{dt}}=\frac{aL_A\cos\zeta-bL_B\sin\zeta}{\cos\eta}$$

$$\frac{\mathrm{d}\eta}{\mathrm{dt}}=aL_A\sin\zeta+bL_B\cos\zeta$$

$$\frac{\mathrm{d}\zeta}{\mathrm{dt}}=cL_C-\left(\frac{\mathrm{d}\xi}{\mathrm{dt}}\right)\sin\eta$$

$$\frac{\mathrm{d}L_A}{\mathrm{dt}}=N_A-(b-c)L_CL_B$$

$$\frac{\mathrm{d}L_B}{\mathrm{dt}}=N_B-(c-a)L_AL_C$$

$$\frac{\mathrm{d}L_C}{\mathrm{dt}}=N_C-(a-b)L_BL_A$$

Andoyer Set (4)

- Andoyer-like Variable $\left(F, w, v, f, y, x\right)$
- New Ang. Mom. CS $\left(\mathbf{e}_P \quad \mathbf{e}_Q \quad \mathbf{e}_R\right)$ $\mathbf{e}_R \equiv \mathbf{e}_G$
  - F, w, v: Use of Torque $\mathbf{N} = N_A \mathbf{e}_A + N_B \mathbf{e}_B + N_C \mathbf{e}_C$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = N_R, \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{-N_Q}{F \cos v}, \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{N_P}{F}$$

$$N_P \equiv \mathbf{N} \cdot \mathbf{e}_P, \text{ etc.}$$

  - f, y, x: Inverse Relation of Angular Velocity

$$\frac{\mathrm{d}f}{\mathrm{d}t} \mathbf{e}_R - \frac{\mathrm{d}y}{\mathrm{d}t} \mathbf{e}_y + \frac{\mathrm{d}x}{\mathrm{d}t} \mathbf{e}_A = \boldsymbol{\omega} - \boldsymbol{\Omega}'$$

$$\boldsymbol{\Omega}' \equiv \frac{\mathrm{d}w}{\mathrm{d}t} \mathbf{e}_X + \frac{\mathrm{d}v}{\mathrm{d}t} \mathbf{e}_v$$

Andoyer Set (5)

- Rotation of New Ang. Mom. CS

$$\mathbf{\Omega}' \equiv \frac{\mathrm{d}w}{\mathrm{d}t}\mathbf{e}_X + \frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{e}_Q$$

$$\mathbf{e}_X = \mathbf{e}_P \cos v + \mathbf{e}_R \sin v$$

- Angular Velocity of CS Rotation

$$\Omega'_P = -\left(\frac{N_Q}{F}\right), \quad \Omega'_Q = \frac{N_P}{F}, \quad \Omega'_R = -\left(\frac{N_Q}{F}\right)\tan v$$

Andoyer Set (6)

- Eq. of Motion: f, y, x (Q: Derive)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left[c + (b-c)\sin^2 x\right]F + \left(\frac{N_Q}{F}\right)\tan v$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -(b-c)F\cos y\sin x\cos x + \frac{1}{F}\left(N_P\cos f + N_Q\sin f\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left[(a-c) - (b-c)\sin^2 x\right]F\sin y$$

$$+\frac{1}{F}\left[\left(N_Q\cos f - N_P\sin f\right)\cos y + N_Q\tan v\sin y\right]$$

Inertial Set (6)

- Case A<B: New Angle and AM $(\xi, \eta, \zeta, \mathbf{L})$
- Eq. of Motion for AM $\frac{\mathrm{d}\mathbf{L}}{\mathrm{dt}} = \mathbf{N}$
- Another Intermediate CS

$$(\mathbf{e}_D \quad \mathbf{e}_E \quad \mathbf{e}_C)^{\mathrm{T}} = \mathbf{R}_2(\eta)\mathbf{R}_1(\xi)$$

$$\mathbf{e}_D = \begin{pmatrix} \cos\eta \\ \sin\eta\sin\xi \\ -\sin\eta\cos\xi \end{pmatrix}, \mathbf{e}_E = \begin{pmatrix} 0 \\ \cos\xi \\ \sin\xi \end{pmatrix}, \mathbf{e}_C = \begin{pmatrix} \sin\eta \\ -\cos\eta\sin\xi \\ \cos\eta\cos\xi \end{pmatrix}$$

Inertial Set (7)

- Eq. of Motion for Angle (Q: Derive)

$$\frac{\mathrm{d}\xi}{\mathrm{dt}} = \frac{1}{\cos\eta}\left[\left(\frac{a+b}{2}\right)L_D + \left(\frac{a-b}{2}\right)\left(L_D \cos 2\zeta + L_E \sin 2\zeta\right)\right]$$

$$\frac{\mathrm{d}\eta}{\mathrm{dt}} = \left(\frac{a+b}{2}\right)L_E + \left(\frac{a-b}{2}\right)\left(L_D \sin 2\zeta - L_E \cos 2\zeta\right)$$

$$\frac{\mathrm{d}\zeta}{\mathrm{dt}} = cL_C - \left(\frac{\mathrm{d}\xi}{\mathrm{dt}}\right)\sin\eta$$

$$L_D = L_X \cos\eta + \left(L_Y \sin\xi - L_Z \cos\xi\right)\sin\eta, L_E = L_Y \cos\xi + L_Z \sin\xi$$

$$L_C = L_X \sin\eta - \left(L_Y \sin\xi - L_Z \cos\xi\right)\cos\eta$$

Inertial Set (8)

- Case A=B
  - Integral $N_C = 0 \rightarrow L_C = \text{const.}$
  - Reducing Number of Variable

$$L_Z = \frac{L_C - L_X \sin\eta + L_Y \sin\xi \cos\eta}{\cos\xi \cos\eta}$$

  - Again, Complexity 4

$$\left(\xi, \eta, L_X, L_Y\right)$$

Inertial Set (9)

- Case A=B (2)
  - Eq. of Motion for Prec. and Incl. Angle

$$\frac{\mathrm{d}\xi}{\mathrm{dt}} = \frac{aL_D}{\cos\eta}$$

$$\frac{\mathrm{d}\eta}{\mathrm{dt}} = aL_E$$

$$L_D \equiv \frac{L_X}{\cos\eta} - L_C \tan\eta$$

$$L_E \equiv \frac{L_Y}{\cos\xi} + \left(\frac{L_C}{\cos\eta} - L_X \tan\eta \right) \tan\xi$$

  - Rot. Angle Integration $\frac{\mathrm{d}\zeta}{\mathrm{dt}} = cL_C - aL_D \tan\eta$

Inertial Set (10)

- Case A=B (3)
  - Eq. of Motion for Angular Momentum

$$\frac{\mathrm{d}L_X}{\mathrm{dt}} = N_D \cos\eta$$

$$\frac{\mathrm{d}L_Y}{\mathrm{dt}} = N_D \sin\xi \sin\eta + N_E \cos\xi$$

$$r_D = \mathbf{r} \cdot \mathbf{e}_D, \text{etc.}$$

$$N_D = -(C-A)\sum \frac{3\mu}{r^5} r_C r_E, N_E = +(C-A)\sum \frac{3\mu}{r^5} r_C r_D$$

Quasi-Element of Rotation

- (Quasi) Element = Const/Linear F. for No Pert.
- Osculating Element = Variable Reducing to Q-Element when Unperturbed
  - OE Becomes Non-Linear F. of Time when Perturbed
- Ex. of Osculating Quasi-Element
  - G, h, I + Max. of tan J + Argument of EF + Linear Part of g

$$\left(G, h, I, \tau \equiv \tan J_0 = \frac{k}{\sqrt{n}}, u, s \right)$$

Element → Variable

- To Andoyer Variable (Q: Confirm)

$$g = s + \sqrt{(1+n)(1+\tau^2)}\,\mathrm{pn}(u;n\,|\,m)$$

$$J = \tan^{-1}\left[\frac{\tau\,\mathrm{qn}(u;n\,|\,m)}{\mathrm{dn}(u\,|\,m)}\right]$$

$$\ell = \frac{\pi}{2} - \mathrm{am}(u\,|\,m) + \tan^{-1}\left[\frac{n\,\mathrm{sn}(u\,|\,m)\,\mathrm{cn}(u\,|\,m)}{\sqrt{1+n}\left(1+\sqrt{1+n}\right)\left(1-n\,\mathrm{sn}^2(u\,|\,m)\right)}\right]$$

$$n \equiv \frac{a-b}{b-c} \qquad m \equiv n\tau^2 \qquad \mathrm{qn}(u;n\,|\,m) \equiv \sqrt{1+n\,\mathrm{sn}^2(u\,|\,m)}$$

Element → Variable (2)

- Case of Earth
  - Can be Approximated as m=0 (Q: Confirm)

$$g \cong s + \sqrt{1+\tau^2}u + \tan^{-1}\left(\frac{n\sin u\cos u}{1+\sqrt{1+n}+n\sin^2 u}\right)$$

$$J \cong \tan^{-1}\left(\tau\sqrt{1+n\sin^2 u}\right)$$

$$\ell \cong \frac{\pi}{2} - u + \tan^{-1}\left[\frac{n\sin u\cos u}{\sqrt{1+n}\left(1+\sqrt{1+n}\right)\left(1-n\sin^2 u\right)}\right]$$

Variable → Element

- From Andoyer Variable (Q: Show)

$$\tau = (\tan J)\sqrt{\frac{1+n\sin^2 \ell}{1+n\left(1+\tan^2 J\cos^2 \ell\right)}}$$

$$u = F\left(\frac{\pi}{2} - \ell + \tan^{-1}\left(\frac{n\sin\ell\cos\ell}{1+\sqrt{1+n}+n\sin^2 \ell}\right); m\right)$$

$$s = g - \sqrt{(1+n)\left(1+\tau^2\right)}\,\mathrm{pn}\left(u; n \mid m\right)$$

Variable → Element (2)

- Case of Earth: Again m=0

$$\tau \cong (\tan J)\sqrt{\frac{1+n\sin^2\ell}{1+n+\tan^2 J\cos^2\ell}}$$

$$u \cong \frac{\pi}{2}-\ell+\tan^{-1}\left(\frac{n\sin\ell\cos\ell}{1+\sqrt{1+n}+n\sin^2\ell}\right)$$

$$s \cong g-\sqrt{1+\tau^2}u+\tan^{-1}\left(\frac{n\sin u\cos u}{1+\sqrt{1+n}+n\sin^2 u}\right)$$

Part. Deriv. by Element

- PD of Andoyer Var., J (Q: Derive)
  - Other PD = 0

$$\left(\frac{\partial J}{\partial \tau}\right)_{G_0,h_0,I_0,u,s} = \frac{\mathrm{qn}(u;n|m)}{\mathrm{dn}(u|m)} + n\left(\frac{1+\tau^2}{1-m}\right)\frac{\mathrm{sn}(u|m)\mathrm{cn}(u|m)}{\mathrm{qn}(u;n|m)}$$

$$\times\left[(1-m)u - \mathrm{en}(u|m) + m\frac{\mathrm{sn}(u|m)\mathrm{cn}(u|m)}{\mathrm{dn}(u|m)}\right]$$

$$\left(\frac{\partial J}{\partial \mathrm{u}}\right)_{G_0,h_0,I_0,\tau,s} = n\tau\left(1+\tau^2\right)\frac{\mathrm{sn}(u|m)\mathrm{cn}(u|m)}{\mathrm{qn}(u;n|m)}$$

PD by Element (2)

- PD of Andoyer Var., ℓ (Q: Derive)
  - Other PD = 0

$$\left(\frac{\partial \ell}{\partial \tau}\right)_{G_0,h_0,I_0,u,s} = \frac{-n\tau\sqrt{1+n}\,\mathrm{dn}(u\,|\,m)}{(1-m)\,\mathrm{qn}^2(u;n\,|\,m)} \times\left[\frac{(1-m)u-\mathrm{en}(u\,|\,m)}{m}+n^2\frac{\mathrm{sn}(u\,|\,m)\,\mathrm{cn}(u\,|\,m)}{\mathrm{dn}(u\,|\,m)}\right]$$

$$\left(\frac{\partial \ell}{\partial \mathrm{u}}\right)_{G_0,h_0,I_0,\tau,s} = \frac{-\sqrt{1+n}\,\mathrm{dn}(u\,|\,m)}{\mathrm{qn}^2(u;n\,|\,m)}$$

PD by Element (3)

- PD of Andoyer Var., g (Q: Derive)
  - Other PD = 0

$$\left(\frac{\partial g}{\partial \tau}\right)_{G_0,h_0,I_0,u,s} = \tau\sqrt{\frac{1+n}{1+\tau^2}}\left[\mathrm{en}(u\,|\,m)-\left(\frac{m}{1-m}\right)\frac{\mathrm{sn}(u\,|\,m)\,\mathrm{cn}(u\,|\,m)}{\mathrm{dn}(u\,|\,m)}\right]$$

$$\left(\frac{\partial g}{\partial \mathrm{u}}\right)_{G_0,h_0,I_0,\tau,s} = \frac{\sqrt{(1+n)(1+\tau^2)}}{\mathrm{qn}^2(u;n\,|\,m)}$$

$$\left(\frac{\partial g}{\partial \mathrm{s}}\right)_{G_0,h_0,I_0,\tau,u} = 1$$

Eq. of Element

- G, h, I: Same as in Andoyer Set
- Eq. of Time Variation of Other Element
  - Q: Derive (May Be Tough)

$$\frac{\mathrm{d}\tau}{\mathrm{dt}} = \nu_\tau, \quad \frac{\mathrm{d}u}{\mathrm{dt}} = \sqrt{\frac{1+n}{1+\tau^2}}\left(b-c\right)G + \nu_u, \quad \frac{\mathrm{d}s}{\mathrm{dt}} = cG + \nu_s$$

$$\nu_\tau = \frac{1+\tau^2}{\tau}\left[\frac{m\,\mathrm{sn}\left(u\,|\,m\right)\mathrm{cn}\left(u\,|\,m\right)}{\sqrt{1+n}}\left(\frac{\Omega_M}{\sin J}\right) + \frac{\mathrm{dn}\left(u\,|\,m\right)}{\mathrm{qn}\left(u;n\,|\,m\right)}\left(\frac{N_G}{G}\cot J - \Omega_J\right)\right]$$

$$\mathrm{qn}\left(u;n\,|\,m\right) \equiv \sqrt{1+n\,\mathrm{sn}^2\left(u\,|\,m\right)} \qquad \sin J = \frac{\tau}{\sqrt{1+\tau^2}}\mathrm{qn}\left(u;n\,|\,m\right)$$

Eq. of Element (2)

$$\nu_u = \left(\frac{n\tau\left(1+\tau^2\right)\mathrm{fn}\left(u\,|\,m\right)}{\mathrm{qn}\left(u;n\,|\,m\right)}\right)\left[\Omega_J - \left(\frac{N_G}{G}\right)\cot J\right]$$
$$-\left[\frac{\mathrm{qn}^2\left(u;n\,|\,m\right)+n\left(1+\tau^2\right)\mathrm{sn}\left(u\,|\,m\right)\mathrm{cn}\left(u\,|\,m\right)\mathrm{fn}\left(u\,|\,m\right)}{\sqrt{1+n}}\right]\left(\frac{\Omega_M}{\sin J}\right)$$

$$\nu_s = -\left(\Omega_G + \Omega_M \cot J\right) - \left(\frac{\nu_u\sqrt{\left(1+n\right)\left(1+\tau^2\right)}}{\mathrm{qn}^2\left(u;n\,|\,m\right)}\right)$$
$$-\tau\nu_\tau\sqrt{\frac{1+n}{1+\tau^2}}\left[\mathrm{pn}\left(u;-m\,|\,m\right)+\frac{n\left(1+\tau^2\right)\mathrm{fn}\left(u\,|\,m\right)}{\mathrm{qn}^2\left(u;n\,|\,m\right)\mathrm{dn}\left(u\,|\,m\right)}\right]$$

$$\mathrm{fn}\left(u\,|\,m\right) \equiv \mathrm{dn}\left(u\,|\,m\right)\left[u - \mathrm{pn}\left(u;-m\,|\,m\right)\right]$$

Another Element

- Another Ex. of Element $\left(\mathbf{L}, \tau, u, s\right)$
  - Angular Momentum in Inertial CS, **L**
  - Other 3 are the Same as Previous
- Q: Derive
  - Transf. Formula with Suitable Variable Set
  - Analytic Expr. of Partial Derivative
  - Eq. of Time Variation of Element

Encke's Method

- Integrate Difference from Approx. Sol.
- Case of Euler Set
  - Approx. Sol. = Unif. Rot. Around C-Axis

$$\omega_A \cong \omega_B \cong 0, \omega_C \cong \omega_0, \psi \cong \psi_0, \theta \cong \theta_0, \phi \cong \phi_0 + \omega_0 t$$

  - Transf. to Small Non-Dimensional Var.

$$m_A \equiv \frac{\omega_A}{\omega_0}, m_B \equiv \frac{\omega_B}{\omega_0}, m_C \equiv \frac{\omega_C}{\omega_0} - 1$$

$$\Delta\psi \equiv \psi - \psi_0, \Delta\theta \equiv \theta - \theta_0, \Delta\phi \equiv \phi - \phi_0 - \sigma$$

$$\sigma \equiv \omega_0 t$$

Encke's Method (2)

- Transf. Eq. of Motion = Polar Motion Eq.

$$\frac{\mathrm{d}m_A}{\mathrm{d}\sigma} = \frac{aN_A}{\omega_0^2} - \alpha\left(1+m_C\right)m_B$$

$$\frac{\mathrm{d}m_B}{\mathrm{d}\sigma} = \frac{bN_B}{\omega_0^2} - \beta\left(1+m_C\right)m_A$$

$$\frac{\mathrm{d}m_C}{\mathrm{d}\sigma} = \frac{cN_C}{\omega_0^2} - \gamma m_A m_B$$

$$\frac{\mathrm{d}\Delta\psi}{\mathrm{d}\sigma} = \frac{m_A \sin\phi + m_B \cos\phi}{\sin\theta}$$

$$\frac{\mathrm{d}\Delta\theta}{\mathrm{d}\sigma} = m_A \cos\phi - m_B \sin\phi$$

$$\frac{\mathrm{d}\Delta\phi}{\mathrm{d}\sigma} = m_C - \left(\frac{\mathrm{d}\Delta\psi}{\mathrm{d}\sigma}\right)\cos\theta$$

$$\psi \equiv \psi_0 + \Delta\psi \qquad \theta \equiv \theta_0 + \Delta\theta \qquad \phi \equiv \phi_0 + \sigma + \Delta\phi$$

Encke's Method (3)

- Case of Serret Canonical Variable
  - Approx. Sol.: Torque-Free Rot. for A=B

$$L = L_0, G = G_0, H = H_0, \ell = \ell_0 + n_\ell t, g = g_0 + n_g t, h = h_0$$

  - Redefining Mean Motion $n_\ell \equiv -\left(\frac{a+b}{2} - c\right) L_0,\ n_g \equiv \left(\frac{a+b}{2}\right) G_0$
  - Transformation to Small Variables

$$\Delta L \equiv L - L_0, \Delta G \equiv G - G_0, \Delta H \equiv H - H_0, \Delta h \equiv h - h_0$$

$$\Delta \ell \equiv \ell - \ell_0 - n_\ell t, \Delta g \equiv g - g_0 - n_g t$$

Encke's Method (4)

- Transfrormed Eq. of Motion (Q: Derive)

$$\frac{\mathrm{d}\Delta G}{\mathrm{d}t} = \left(\frac{\partial V}{\partial g}\right)$$

$$\frac{\mathrm{d}\Delta H}{\mathrm{d}t} = \left(\frac{\partial V}{\partial h}\right)$$

$$\frac{\mathrm{d}\Delta h}{\mathrm{d}t} = -\left(\frac{\partial V}{\partial H}\right)$$

$$\frac{\mathrm{d}\Delta L}{\mathrm{d}t} = -\left(\frac{a-b}{2}\right)_n \left(G^2 - L^2\right)\sin 2\ell + \left(\frac{\partial V}{\partial \ell}\right)$$

$$\frac{\mathrm{d}\Delta \ell}{\mathrm{d}t} = -\left(\frac{a+b}{2} - c\right)\Delta L + \left(\frac{a-b}{2}\right) L\cos 2\ell - \left(\frac{\partial V}{\partial L}\right)$$

$$\frac{\mathrm{d}\Delta g}{\mathrm{d}t} = \left(\frac{a+b}{2}\right)\Delta G - \left(\frac{a-b}{2}\right) G\cos 2\ell - \left(\frac{\partial V}{\partial G}\right)$$

$$L = L_0 + \Delta L, G = G_0 + \Delta G, H = H_0 + \Delta H, h = h_0 + \Delta h$$
$$\ell = \ell_0 + n_\ell t + \Delta \ell, g = g_0 + n_g t + \Delta g$$

Encke's Method (5)

- Case of Another Element
  - Approx. Sol.: General Torque-Free Rotation

$$\mathbf{L} \cong \mathbf{L}_0, \tau \cong \tau_0, u \cong u_0 + n_u t, s \cong s_0 + n_s t$$

  - Mean Motion

$$n_u \equiv \frac{G_0\sqrt{(a-c)(b-c)}}{\sqrt{1+\tau_0^2}},\ n_s \equiv cG_0 \quad G_0 \equiv |\mathbf{L}_0|$$

  - Transformation to Small Variable

$$\Delta\mathbf{L} \equiv \mathbf{L} - \mathbf{L}_0, \Delta\tau \equiv \tau - \tau_0$$

$$\Delta u \equiv u - u_0 - n_u t, \Delta s \equiv s - s_0 - n_s t$$

Encke's Method (6)

- Transformed Eq. of Motion (Q: Derive)

$$\frac{\mathrm{d}\Delta\mathbf{L}}{\mathrm{dt}}=\mathbf{N},\quad \frac{\mathrm{d}\Delta\tau}{\mathrm{dt}}=v_\tau,\quad \frac{\mathrm{d}\Delta s}{\mathrm{dt}}=c\Delta G+\Omega_s$$

$$\frac{\mathrm{d}\Delta u}{\mathrm{dt}}=\sqrt{\frac{(a-c)(b-c)}{1+\tau^2}}\left[\Delta G+\frac{G_0(\tau+\tau_0)\Delta\tau}{\sqrt{1+\tau_0^2}\left(\sqrt{1+\tau^2}+\sqrt{1+\tau_0^2}\right)}\right]+\Omega_u$$

$$\Delta G\equiv\frac{(\mathbf{L}+\mathbf{L}_0)\cdot\Delta\mathbf{L}}{G+G_0}\qquad G\equiv\sqrt{\mathbf{L}^2},\quad G_0\equiv\sqrt{\mathbf{L}_0^2}$$

$$\mathbf{L}\equiv\mathbf{L}_0+\Delta\mathbf{L},\tau\equiv\tau_0+\Delta\tau,u\equiv u_0+n_ut+\Delta u,s\equiv s_0+n_st+\Delta s$$

Fast Rotation Difficulty

- Ordinary Approach
  - Case A=B: Rot. Angle Appears Explicitly
  - Step Size of Integration Must Be Small
- Solution (Case: A=B)
  - Poisson Approx. + Oppolzer Term
  - Inertial Set
- Or Fast Integrator, Fast Computer, …

Poisson Approximation

- Assuming
  - No.1: A=B
  - No.2: Rot. Axis = AM Axis = Figure Axis

$$N_C = 0 \rightarrow \frac{\mathrm{d}L_C}{\mathrm{dt}} = 0 \rightarrow L_C = C\omega_0$$

$$L_A = L_B = 0 \rightarrow \mathbf{L} = C\omega_0 \mathbf{e}_C$$

- Approx. EoM (Q: Derive)

$$\frac{\mathrm{d}\mathbf{e}_C}{\mathrm{dt}} = \frac{\mathbf{N}}{C\omega_0}$$

$$\frac{\mathrm{d}\psi_P}{\mathrm{dt}} = \frac{N_X \cos\psi_P + N_Y \sin\psi_P}{C\omega_0 \sin\theta_P}$$

$$\frac{\mathrm{d}\theta_P}{\mathrm{dt}} = \frac{N_X \sin\psi_P - N_Y \cos\psi_P}{C\omega_0 \cos\theta_P}$$

$$\frac{\mathrm{d}\phi_P}{\mathrm{dt}} = \omega_0 - \left(\frac{\mathrm{d}\psi_P}{\mathrm{dt}}\right)\cos\theta_P$$

Oppolzer Term

- Difference Between Rot. and Figure Axis
  - Corr. of Poisson Approx. $\theta_Q \equiv \theta - \theta_P, \psi_Q \equiv \psi - \psi_P$
- Approx. Eq. of Motion of Correction Term

$$\frac{\mathrm{d}\theta_Q}{\mathrm{dt}} = \frac{-1}{C\omega_C \sin\theta_P}\left[\left(\frac{\partial^2 U}{\partial\theta\,\partial\psi}\right)_P \theta_Q + \left(\frac{\partial^2 U}{\partial\psi^2}\right)_P \psi_Q\right] + \frac{A}{C\omega_C}\left[2\left(\frac{\mathrm{d}\psi_P}{\mathrm{dt}}\right)\left(\frac{\mathrm{d}\theta_P}{\mathrm{dt}}\right)\cos\theta_P + \left(\frac{\mathrm{d}^2\psi_P}{\mathrm{dt}^2}\right)\sin\theta_P\right]$$

Oppolzer Term (2)

- Approx. EoM of Corr. Term (2)

$$\frac{\mathrm{d}\psi_Q}{\mathrm{dt}} = \frac{1}{C\omega_C \sin^2\theta_P}\left[\left(\frac{\partial^2 U}{\partial \theta^2}\right)_P \theta_Q + \left(\frac{\partial^2 U}{\partial\theta\,\partial\psi}\right)_P \psi_Q\right]$$

$$+\frac{1}{\sin\theta_P}\left[\left(\frac{A}{C\omega_C}\right)\left\{\left(\frac{\mathrm{d}\psi_P}{\mathrm{dt}}\right)^2 \sin\theta_P \cos\theta_P - \left(\frac{\mathrm{d}^2\theta_P}{\mathrm{dt}^2}\right)\right\}\right.$$

$$\left. - \left\{\left(\frac{\mathrm{d}\psi_P}{\mathrm{dt}}\right)\cos\theta_P\right\}\theta_Q\right]$$

Integration Method for Rotational Motion

- Method for General 1st Order ODE
  - Method for Special 2nd Order ODE (Störmer-Cowell, Hermite, etc.) Can Not Be Used
- General Numerical Integrator
  - Single Step: Runge-Kutta, Extrapolation
  - Multi-step: Adams
- Symplectic Integrator
  - Applicable to Serret Canonical Variable

9. Non-Rigid Effect

- Ex.: Rotation of Non-Rigid Earth
- Rigid → Non-Rigid
  - Main CS: Principal CS → Tisserand Mean CS
  - Eq. of Motion: Euler → Liouville
  - Mom. Inertia: Diagonal → Non-Diagonal
  - Intern. Structure: Single Layer → Multi-Layer
  - Difficulty: Treatment of Fluid (Poincaret Approx.)
- Non-Rigid Nutation: Linear Response Theory

Non-Rigid Earth

- Multi-Layer: Crust, Mantle, Fluid & Inner Core
  - No Longer the Same Rotation → Chandler Motion
- Elastic Response: Earth Tide, Pole Tide
  - Prop. Coeff.: Love Number, Shida Number, …
- Dissipation: Unresolved Secular Slow-Down
- Excitation Mechanism of Damping Motion
  - Polar Motion, Free Core Nutation, etc.
- Effect of Fulid: Ocean, Atmosphere, …

Chandler Motion

- Free Nutation (= Free Polar Motion) of Earth
- Circular Oscill. of A- and B-Axis of Ang. Vel.
  - Euler's Prediction: P = 306〜325 day
- Observed Value (Chandler, AJ, 1891)
  - P ~ 427 day (Also Annual Component)
- Big Issue: Why Different?
  - Newcomb (1891) Non-Rigidity Concerns
  - International Lat. Obs. (ILO): Mizusawa, …

Tisserand Mean CS

- One of Body-Fixed CS for Non-Rigid Body
- Definition by Tisserand
  - AM Expr. in a Certain Rigidly-Rotating CS

$$\mathbf{L} = \int \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x} + \mathbf{v}) \rho(\mathbf{x}) \, \mathrm{d}^3\mathbf{x} = \mathbf{I}\boldsymbol{\omega} + \mathbf{h}$$

  - **h**: Ang. Mom. Due to Internal Motion
  - Define CS such that **h**=0 Always
  - Tisserand Mean Axis = Z-Axis of TMCS

Liouville Eq. of Motion

- Ang. Mom. Conservation $\frac{d\mathbf{L}}{dt} = \mathbf{N}$
- Tisserand Mean CS
  - **I**: Depends on Time and Angular Velocity

$$\mathbf{I}\left(\frac{D\boldsymbol{\omega}}{Dt}\right) = \mathbf{N} - \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} - \left(\frac{D\mathbf{I}}{Dt}\right)\boldsymbol{\omega}$$

- Eq. of Rotational Motion by Liouville
  - Time/AV-Dependency of **I** Must Be Modeled

Liouville Equation (2)

- Separation of **I** $\mathbf{I}(\boldsymbol{\omega}, t) = \mathbf{I}_0 + \Delta\mathbf{I}(\boldsymbol{\omega}, t)$
  - Const. & Diagonal + Time-Var. & Non-Diag.
- Euler-Set-like Expression
  - Assuming Smallness of Non-Diag. Part

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{Dt}} = \mathbf{W}_0 + \Delta\mathbf{W}$$

$$\mathbf{W}_0 \equiv \mathbf{I}_0^{-1}\left(\mathbf{N} - \boldsymbol{\omega} \times \mathbf{I}_0\boldsymbol{\omega}\right) \qquad \Delta\mathbf{W} \equiv -\mathbf{I}_0^{-1}\left[\Delta\mathbf{I}\mathbf{W}_0 + \left(\frac{\mathrm{D}\Delta\mathbf{I}}{\mathrm{Dt}}\right)\boldsymbol{\omega} + \boldsymbol{\omega} \times \Delta\mathbf{I}\boldsymbol{\omega}\right]$$

Inertial Set (11)

- Variable $(\psi, \theta, \phi, \mathbf{L})$
- OK for Non-Rigidity

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{dt}} = \mathbf{N}$$

$$\boldsymbol{\omega} = \mathbf{I}^{-1}\mathbf{L}$$

$$\frac{\mathrm{d}\psi}{\mathrm{dt}} = \omega_Z - \left(\frac{\mathrm{d}\phi}{\mathrm{dt}}\right)\cos\theta$$

$$\frac{\mathrm{d}\theta}{\mathrm{dt}} = \omega_X \cos\psi + \omega_Y \sin\psi$$

$$\frac{\mathrm{d}\phi}{\mathrm{dt}} = \frac{\omega_X \sin\psi - \omega_Y \cos\psi}{\sin\theta}$$

$$(\mathbf{e}_A \quad \mathbf{e}_B \quad \mathbf{e}_C) = \mathbf{R}_3(\phi)\mathbf{R}_1(\theta)\mathbf{R}_3(\psi)$$

- Torque in Inertial CS

$$N_X = \frac{-3\mu}{r^5}\left[r_X r_Y I_{ZX} - r_Z r_X I_{XY} + \left(r_Y^2 - r_Z^2\right) I_{YZ} + r_Y r_Z \left(I_{ZZ} - I_{YY}\right)\right]$$

Inertial Set (12)

- Variable $(\xi, \eta, \zeta, \mathbf{L})$
- 1-2-3 Convention

Avoiding Difficulty

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{dt}} = \mathbf{N} \qquad \boldsymbol{\omega} = \mathbf{I}^{-1}\mathbf{L}$$

$$\begin{pmatrix}\mathbf{e}_A & \mathbf{e}_B & \mathbf{e}_C\end{pmatrix} = \mathbf{R}_3(\zeta)\mathbf{R}_2(\eta)\mathbf{R}_1(\xi)$$

$$\frac{\mathrm{d}\xi}{\mathrm{dt}} = \omega_X - \left(\frac{\mathrm{d}\zeta}{\mathrm{dt}}\right)\sin\eta$$

$$\frac{\mathrm{d}\eta}{\mathrm{dt}} = \omega_Y \cos\xi + \omega_Z \sin\xi$$

$$\frac{\mathrm{d}\zeta}{\mathrm{dt}} = \frac{-\omega_Y \sin\xi + \omega_Z \cos\xi}{\cos\eta}$$

$$N_X = \frac{-3\mu}{r^5}\left[r_X r_Y I_{ZX} - r_Z r_X I_{XY} + \left(r_Y^2 - r_Z^2\right) I_{YZ} + r_Y r_Z \left(I_{ZZ} - I_{YY}\right)\right]$$

Mom. Inertia Correction

- Correction of $\mathbf{I}$ Due To Non-Rigidity

  $\mathbf{I} = \mathbf{I}_0 + \Delta\mathbf{I}$ $\Delta\mathbf{I} = \Delta_S\mathbf{I} + \Delta_P\mathbf{I} + \Delta_F\mathbf{I}$
- Non-Rigid Terms $\Delta\mathbf{I}$
  - Body Tide: Due to Tidal Force $\Delta_S\mathbf{I}$
  - Pole Tide: Due to Centrifugal Force $\Delta_P\mathbf{I}$
  - Interal Friction: Due to Other Layer $\Delta_F\mathbf{I}$
- Spherical Harmonics Also Change

Body Tide = Earth Tide

- Corr. Due To External Tidal Force
  - Assuming Dipole Var. (P_2 Mode)

$$\Delta_E \mathbf{I}(t) = \left(\frac{k_2 R^5}{3G}\right) \sum \frac{m}{r^5} \left(\mathbf{r} \otimes \mathbf{r} - \frac{r^2}{3}\mathbf{1}\right)_{t-\tau_2}$$

  - k_2 : Love Number
  - R: Equatorial Radius of Finite Body
  - τ_2 : Time Lag → Energy Dissipation

Pole Tide

- Corr. Due To Centrifugal Force
  - Again Assuming Dipole Variation

$$\Delta_P \mathbf{I}(t) = \left(\frac{k_2 R^5}{3G}\right)\left(\boldsymbol{\omega} \otimes \boldsymbol{\omega} - \frac{\omega^2}{3}\mathbf{1}\right)_{t-\tau_2}$$

  - Again (Small) Dissipation

Corr. Spherical Harmonics

- MacCullagh's Formula → Torque Eval.

$$GMR^2C_{20} = \frac{I_{AA} + I_{BB}}{2} - I_{CC}$$

$$GMR^2C_{21} = -I_{AC}$$

$$GMR^2S_{21} = -I_{BC}$$

$$GMR^2C_{22} = \frac{I_{BB} - I_{AA}}{4}$$

$$GMR^2S_{22} = -I_{AB}$$

$$I_{JK} \equiv \mathbf{e}_J \cdot \mathbf{I}\mathbf{e}_K$$

Poincare Theory

- Two Layer: Mantle + Fluid Core
- Total Kin. Energy $2T \equiv \boldsymbol{\omega}_M \mathbf{I}_M \boldsymbol{\omega}_M + \boldsymbol{\omega}_C \mathbf{I}_C \boldsymbol{\omega}_C$
- Diff. Ang. Vel. $\Delta\boldsymbol{\omega}_C \equiv \boldsymbol{\omega}_C - \boldsymbol{\omega}_M$
- Eq. of Motion
  - Derived from Variational Pr.

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial T}{\partial \boldsymbol{\omega}_M}\right) + \boldsymbol{\omega}_M \times \left(\frac{\partial T}{\partial \boldsymbol{\omega}_M}\right) = \mathbf{N}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial T}{\partial \Delta\boldsymbol{\omega}_C}\right) - \Delta\boldsymbol{\omega}_C \times \left(\frac{\partial T}{\partial \Delta\boldsymbol{\omega}_C}\right) = 0$$

Poincare Theory (2)

- Approx. Eq. of Motion

$$\frac{d\left(\mathbf{I}\boldsymbol{\omega}_M + \mathbf{I}_C \Delta\boldsymbol{\omega}_C\right)}{dt} + \boldsymbol{\omega}_M \times \left(\mathbf{I}\boldsymbol{\omega}_M + \mathbf{I}_C \Delta\boldsymbol{\omega}_C\right) = \mathbf{N}$$

$$\frac{d\left(\mathbf{I}_C\left(\boldsymbol{\omega}_M + \Delta\boldsymbol{\omega}_C\right)\right)}{dt} - \Delta\boldsymbol{\omega}_C \times \mathbf{I}_C \boldsymbol{\omega}_M = 0$$

$$\mathbf{I} \equiv \mathbf{I}_M + \mathbf{I}_C$$

Poincare Theory (3)

- Rotational Symmetry $A_M = B_M, A_C = B_C$
- Two Integrals Exist
- Freedom 4 = 4 Eigen Freq. of Free Rot.
  - SA: Spin-Axis Mode = Unif. Rot.
  - TO: Tilt-Over Mode = Unif. Rot. of Core
  - CW: Chandler Wobble, P~(1-1/400)days
  - FCN: Free Core Nutation, P~431days

10. Application

- Basic 1: Earth Rotation
  - Application: Mars
- Basic 2: Physical Libration of Moon
  - Application: Mercury
- Others
  - Asteroid: Not-So-Well-Known
  - Artificial Satellite: Controlled Rotation

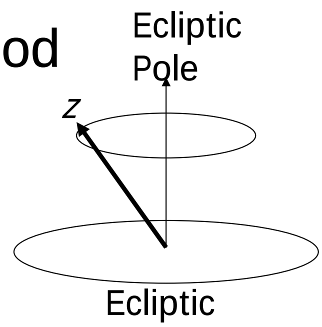
Earth Rotation

- Important: Basis of Coord. Transf. Between Terrestrial and Celestial CS
- Sidereal Motion **S**: Rotation Angle, UT1
- Motion of Figure Axis
  - Nearly-Diurnal: Polar Motion **W**
  - Other: Precession **P** + Nutation **N**
- Matrix Representation

$$\mathbf{R} = \mathbf{WSNP}$$

Precession/Nutation

- Non-Nearly Diurnal Figure Axis Motion
- 2 Component in Ecliptic CS
  - Longitude, Obliquity
- Precession=Very Long Period
  - 50 arcsec/y, P~26000y
- Nutation＝Other Periods
  - 18.6y, 0.5y, 9.3y, …
- Shifting to New Model



Precession

- Discovery: Hipparcus (~150BC)
- Old Model: IAU1976
  - Lieske et al. (1977, A&A)
  - Dynamical Part: Newcomb
  - + Correction in Planetary Mass
  - Adding Geodesic Precession
- Theory in Ecliptic CS
- Representation in Equatorial CS

Oppolzer Represent.

$$\mathbf{P} = \mathbf{R}_{323}(-\zeta_A, \theta_A, -z_A)$$

- 3-Angle Representation in Equatorial CS

$$\begin{pmatrix} \zeta_A \\ \theta_A \\ z_A \end{pmatrix} = \begin{pmatrix} 2306.2181 \\ 2004.3109 \\ 2306.2181 \end{pmatrix} T + \begin{pmatrix} 0.30188 \\ -0.42665 \\ 1.09468 \end{pmatrix} T^2 + \begin{pmatrix} 0.017998 \\ -0.041833 \\ 0.018203 \end{pmatrix} T^3$$

  - Unit: arcsecond
  - T =(JD-2451545.0)/36525

Nutation

- Discovery: Bradley (1747)
- Old Model: IAU1980
  - Seidelmann et al. (1981, CM)
  - Rigid Earth Solution: Kinoshita (1977, CM)
  - Non-Rigid Effect: Wahr (1981, GJRAS)
- Mean Obliquity (Lieske et al. 1977)

$$\varepsilon_A = 23°26'21''.448 - 46''.8150T - 0''.00059T^2 + 0''.001813T^3$$

Nutation (2)

- Matrix Representation

$$\mathbf{N} = \mathbf{R}_{131}\left(\varepsilon_A, -\Delta\psi, -\left(\varepsilon_A + \Delta\varepsilon\right)\right)$$

- Nutation in Longitude $\Delta\psi$
- Nutation in Obliquity $\Delta\varepsilon$
- Analytic Expression

$$\begin{pmatrix} \Delta\psi \\ \Delta\varepsilon \end{pmatrix} = \sum_k \begin{pmatrix} \psi_k \sin A_k \\ \varepsilon_k \cos A_k \end{pmatrix}, \quad A_k = \sum_{j=1}^{5} n_j \Omega_j$$

Delauney Angle

- 5 Major Angle in Nutation Theory
  - Mean Anomaly of Moon ℓ
  - Mean Anomaly of Sun ℓ'
  - Mean Argument of Latitude of Moon F
  - Mean Elongation of Moon from Sun D
  - Mean Longitude of Node of Moon Ω
- Other Angle

$L = F + \Omega$ $\quad L' = L - D$ $\quad \varpi' = L - D - \ell'$

Delauney Angle (2)

- IERS Convention 2003 (2005)

$$\ell = 134.96340251° + 1717915923.2178''T + 31.8792''T^2 + 0.051635''T^3 - 0.00024470''T^4$$

$$\ell' = 357.52910918° + 129596581.0481''T - 0.5532''T^2 + 0.000136''T^3 - 0.00001149''T^4$$

$$F = 93.27209062° + 1739527262.8478''T - 12.7512''T^2 - 0.001037''T^3 + 0.00000417''T^4$$

$$D = 297.85019547° + 160296160.2090''T - 6.3706''T^2 + 0.006593''T^3 - 0.00003169''T^4$$

$$\Omega = 125.04455501° - 6962890.5431''T + 7.4722''T^2 + 0.007702''T^3 - 0.00005939''T^4$$

Main Term of Nutation

- IERS Convention 2003, Unit: arcsec

$$\begin{pmatrix}\Delta\psi \\ \Delta\varepsilon\end{pmatrix} = \begin{pmatrix}-17.206\sin\Omega \\ 9.205\cos\Omega\end{pmatrix} + \begin{pmatrix}-1.317\sin 2L' \\ 0.573\cos 2L'\end{pmatrix} + \begin{pmatrix}0.207\sin 2\Omega \\ -0.090\cos 2\Omega\end{pmatrix}$$

$$+ \begin{pmatrix}-0.228\sin 2L \\ 0.098\cos 2L\end{pmatrix} + \begin{pmatrix}0.148\sin \ell' \\ 0.007\cos \ell'\end{pmatrix} + \begin{pmatrix}0.071\sin \ell \\ -0.001\cos \ell\end{pmatrix}$$

$$+ \begin{pmatrix}-0.052\sin(2L'+\ell') \\ 0.022\cos(2L'+\ell')\end{pmatrix} + \begin{pmatrix}-0.030\sin(2L+\ell) \\ 0.013\cos(2L+\ell)\end{pmatrix}$$

$$+ \begin{pmatrix}0.022\sin(2L'-\ell') \\ -0.010\cos(2L'-\ell')\end{pmatrix} \cdots$$

Sidereal Rotation

- Almost Uniform, Nearly-Diurnal Rotation
  - Ω_0 = 7.2921150(1) x 10^{-5} radian/s
- Rot. Speed = 360 deg/1 Sidereal Day

  ~ 365.2422.../366.2422... rev./day
- Greenwich App. Sidereal T. (GAST) = Θ

$$\mathbf{S} = \mathbf{R}_3(\Theta)$$

Deviation from Unif. Rot.

- UTC → UT1 → GMST → GAST
  - DUT1 = UT1-UTC: Unpredictable
  - GMST = $GMST_0$ + r UT1 + ...
  - Ratio of Sidereal/Universal Time:
    - r ~1.0027379...
  - GAST = GMST + $\Delta\psi$ cos ε + ...
- Length Of Day (LOD) = $2\pi/\Omega$

Polar Motion (=Wobble)

- Slow Motion of Pole Viewed on Earth
  - (x_p, y_p) Magnitude ~ 0.1 arcsec ~ 30m
  - Periods: Annual, Chandler (~14 Month)
- Unpredictable: To be Monitored

$$\mathbf{W} = \mathbf{R}_2\left(-x_p\right)\mathbf{R}_1\left(-y_p\right)$$

Earth Orientation Param.

- Earth Orientation Parameters (EOP)
  - DUT1, LOD, x_p, $y_{p,}$, Pole Offset
  - Old Name: Earth Rotation Param. (ERP)
- Pole Offset = Error in Prec./Nut. Theory
- Internat'l Earth Rotation Service (IERS)
  - Since 1984, Joint Service of IAU + IUGG
  - URL: http://www.iers.org/

Rotation of Moon

- Libration of Moon
  - = Apparent Motion of Moon's Surface
  - = Optical + Physical
- Optical Libration
  - Cassini's Law + Orb. Motion
- Physical Libration
  - Deviation from Cassini's Law

Cassini's Law

- Cassini, J.D. (Report 1693, Publ. 1730)
- Approximation Law
  - Euler Angle of Moon's Rotation in Ecliptic CS

$$\psi \simeq \Omega, \quad \theta \simeq -\theta_0, \quad \phi \simeq F + \pi$$

  - F, Ω : Delauney Angle
  - Obliquity of Moon's Equator w.r.t. Moon's Orbital Plane (Eckhardt 1981) $\theta_0 = 5753''$

Libration Angle

- Three Libration Angle

  $$\sigma \equiv \psi - \Omega,\ \rho \equiv -\left(\theta - \theta_0\right),\ \tau \equiv \phi + \psi - L - \pi$$

- Anal. Sol.: Eckhardt(1981),Moons(1982)
  - Transf. Avoiding Small Angle Difficulty

    $$\left(p_A \equiv -\sin\theta \sin\phi,\, p_B \equiv -\sin\theta \cos\phi, \tau\right)$$

- Numer. Sol.: DE of NASA/JPL
  - Lagrange Set

DE

- Development Ephemeris (NASA/JPL)
- Also Available at NAOJ/ADAC
  - Fortran Routines + Binary File
  - Latest Ver.: DE408: BC10000-AD10000
  - Providing Sun, Moon, Planets
  - Some Version Support Phys. Libration of Moon
- More Solar System Bodies: HORIZONS
  - http://ssd.jpl.nasa.gov/

Main Term of Physical Libration of Moon

- Moons (1982, CMDA) Unit: arcsec
  - d = JD-2451545.0

$$\begin{aligned}
\tau = {} & 174.27 + 90.69 \sin \ell' - 16.79 \sin \ell + 16.79 \sin(2\ell - 2F) \\
& - 14.29 \sin(55.27° + 0.036378° d) + 9.94 \sin(2\ell - 2D) \\
& - 8.09 \sin(1.68° + 0.529524° d) - 6.74 \cos(\ell - F) \\
& + 4.13 \sin(\ell - 2D) - 3.46 \sin(\ell - D) + 1.65 \sin(2F - 2D) \\
& - 1.39 \sin(\ell - F) - 1.15 \sin(\ell - \ell' - D) + 1.00 \cos F + \cdots
\end{aligned}$$

Main Term of Physical Libration of Moon (2)

$$\begin{pmatrix} p_A \\ p_B \end{pmatrix} = \begin{pmatrix} 5562.01 \sin F \\ 5539.88 \cos F \end{pmatrix} + \begin{pmatrix} 124.48 \sin(\ell - F) \\ -75.40 \cos(\ell - F) \end{pmatrix} + \begin{pmatrix} -74.58 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 9.03 \sin L \\ 9.03 \cos L \end{pmatrix} + \begin{pmatrix} -7.29 \cos L \\ 7.29 \sin L \end{pmatrix} + \begin{pmatrix} 4.68 \cos F \\ -4.70 \sin F \end{pmatrix}$$

$$+ \begin{pmatrix} 2.91 \sin(F - 2D) \\ -3.20 \cos(F - 2D) \end{pmatrix} + \begin{pmatrix} -2.68 \sin(\ell + F - 2D) \\ -1.61 \cos(\ell + F - 2D) \end{pmatrix}$$

$$+ \begin{pmatrix} 1.57 \sin(\ell + F) \\ 0 \end{pmatrix} + \begin{pmatrix} 1.24 \sin(\ell' + F) \\ 1.28 \cos(\ell' + F) \end{pmatrix} + \begin{pmatrix} 1.02 \sin(\ell' - F) \\ -1.09 \cos(\ell' - F) \end{pmatrix} \cdots$$

11. General Relativity

- No Rigid Body ← Velocity < c
  - Non-Rigid-Like Treatment: TMCS, …
- Tetrad: Four Basis Vector in Spacetime
- Free Rotation = Gravitation Only
- Fermi-Walker Transport
  - AM Consev. Law in Curved Spacetime
  - Torque-Free = Fermi-Transport

Dragging of Inertia

- Local Parallel Shift $\neq$ Global Non-Rotation
  - No Colioris Force $\neq$ Rest w.r.t. Quasar
- Fermi Transport
  - Gen. Relativistic Extension of Parallel Shift
  - Fermi-Walker Transp. = Fermi Transp. + Rot.
- Proper CS = Fermi-Transpoted CS

Geodesic Rotation

- Spatial Part of Fermi-Transp. Tetrad
- Special Rel.: Thomas Prec. $\frac{\mathbf{v}\times\mathbf{a}}{c^3}$
- General Rel.
  - Geodesic Precession $\frac{(1+\gamma)\,\mathbf{v}\times\nabla\phi}{c^3}$
    - ~1.92 arcsec/cy
    - De Sitter (1917)
  - Geodesic Nutation (Fukushima 1991) $\frac{\nabla\times\mathbf{g}}{c^3}$
  - Lense-Thirring Effect

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